Application of Derivatives

Question1

Let g(x) = 3f(x/3) + f(3 − x) and f″(x) > 0 for all x∈(0, 3). If g is decreasing in (0, α) and increasing in (α, 3), then 8α is

[27-Jan-2024 Shift 2]

Options:

A.

24

B.

0

C.

18

D.

20

Answer: C

Solution:

$g(x) = 3f\left(\frac{x}{3}\right) + f(3-x)$ and $f''(x) > 0\ \forall x \in (0, 3)$

$\Rightarrow f'(x)$ is increasing function

$g'(x) = 3 \times \frac{1}{3} \cdot f'\left(\frac{x}{3}\right) - f'(3-x)$

$= f'\left(\frac{x}{3}\right) - f'(3-x)$

If g is decreasing in $(0, \alpha)$

$g'(x) < 0$

$f'\left(\frac{x}{3}\right) - f'(3-x) < 0$

$$f'\left(\frac{x}{3}\right) < f'(3-x)$$

$$\Rightarrow \frac{x}{3} < 3 - x$$

$$\Rightarrow x < \frac{9}{4}$$

Therefore $\alpha = \frac{9}{4}$

Then $8\alpha = 8 \times \frac{9}{4} = 18$

Question2

The function $f(x) = 2x + 3(x)^{\frac{2}{3}}, x \in \mathbb{R}$, has

[29-Jan-2024 Shift 2]

Options:

A.

exactly one point of local minima and no point of local maxima

B.

exactly one point of local maxima and no point of local minima

C.

exactly one point of local maxima and exactly one point of local minima

D.

exactly two points of local maxima and exactly one point of local minima

Answer: C

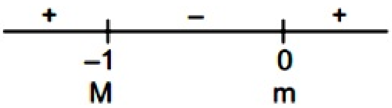
Solution:

$$f(x)=2x+3(x)^{\frac{2}{3}}$$

$$f'(x)=2+2x^{\frac{-1}{3}}$$

$$=2\left(1+\frac{1}{x^{\frac{1}{3}}}\right)$$

$$=2\left(\frac{x^{\frac{1}{3}}+1}{x^{\frac{1}{3}}}\right)$$



So, maxima(M) at $x=-1$ & minima(m) at $x=0$

Question3

The function $f(x)=\dfrac{x}{x^2-6x-16}, x\in\mathbb{R}-\{-2,8\}$

[29-Jan-2024 Shift 2]

Options:

A.

decreases in $(-2, 8)$ and increases in $(-\infty, -2)\cup(8, \infty)$

B.

decreases in $(-\infty, -2)\cup(-2, 8)\cup(8, \infty)$

C.

decreases in $(-\infty, -2)$ and increases in $(8, \infty)$

D.

increases in $(-\infty, -2)\cup(-2, 8)\cup(8, \infty)$

Answer: B

Solution:

$f(x) = \frac{x}{x^2 - 6x - 16}$

Now,

$f'(x) = \frac{-(x^2 + 16)}{(x^2 - 6x - 16)^2}$

$f'(x) < 0$

Thus $f(x)$ is decreasing in

$(-\infty, -2) \cup (-2, 8) \cup (8, \infty)$

Question4

Let g:R→R be a non constant twice differentiable such that g′(1/2) = g′(3/2). If a real valued function f is defined as $f(x) = \frac{1}{2}[g(x) + g(2 - x)]$, then

[30-Jan-2024 Shift 1]

Options:

A.

$f'(x) = 0$ for atleast two x in (0, 2)

B.

$f''(x) = 0$ for exactly one x in (0, 1)

C.

$f'(x) = 0$ for no x in (0, 1)

D.

$f'\left(\frac{3}{2}\right) + f'\left(\frac{1}{2}\right) = 1$

Answer: A

Solution:

$$f'(x) = \frac{g'(x) - g'(2-x)}{2}, f'\left(\frac{3}{2}\right) = \frac{g'\left(\frac{3}{2}\right) - g'\left(\frac{1}{2}\right)}{2} = 0$$

$$\text{Also } f'\left(\frac{1}{2}\right) = \frac{g'\left(\frac{1}{2}\right) - g'\left(\frac{3}{2}\right)}{2} = 0, f'\left(\frac{1}{2}\right) = 0$$

$$\Rightarrow f'\left(\frac{3}{2}\right) = f'\left(\frac{1}{2}\right) = 0$$

$$\Rightarrow \text{ rootsin } \left(\frac{1}{2}, 1\right) \text{ and } \left(1, \frac{3}{2}\right)$$

$$\Rightarrow f''(x) \text{ is zero at least twice in } \left(\frac{1}{2}, \frac{3}{2}\right)$$

Question5

Let $f(x) = (x+3)^2 (x-2)^3$, $x \in [-4, 4]$. If M and m are the maximum and minimum values of f, respectively in $[-4, 4]$, then the value of $M - m$ is :

[30-Jan-2024 Shift 2]

Options:

A.

600

B.

392

C.

608

D.

108

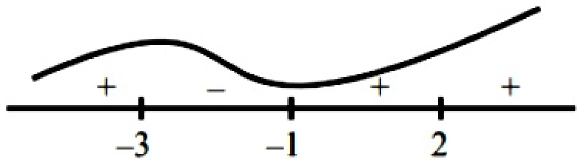
Answer: C

Solution:

$f'(x) = (x+3)^2 \cdot 3(x-2)^2 + (x-2)^3 2(x+3)$

$= 5(x+3)(x-2)^2(x+1)$

$f'(x) = 0,\ x = -3, -1, 2$



$f(-4) = -216$

$f(-3) = 0, f(4) = 49 \times 8 = 392$

$M = 392,\ m = -216$

$M - m = 392 + 216 = 608$

Question6

If $5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 2,\ \forall x \neq 0$ and $y = 9x^2 f(x)$, then y is strictly increasing in :

[1-Feb-2024 Shift 1]

Options:

A.

$\left(0, \frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$

B.

$\left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$

C.

$\left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(0, \frac{1}{\sqrt{5}}\right)$

D.

$\left(-\infty, \frac{1}{\sqrt{5}}\right) \cup \left(0, \frac{1}{\sqrt{5}}\right)$

Answer: B

Solution:

$5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 2, \forall x \neq 0$ ....(1)

Substitute $x \to \frac{1}{x}$

$5f\left(\frac{1}{x}\right) + 4f(x) = \frac{1}{x^2} - 2$ ....(2)

On solving (1) and (2)

$f(x) = \frac{5x^4 - 2x^2 - 4}{9x^2}$

$y = 9x^2 f(x)$

$y = 5x^4 - 2x^2 - 4$ ....(3)

$\frac{dy}{dx} = 20x^3 - 4x$

for strictly increasing

$\frac{dy}{dx} > 0$

$4x(5x^2 - 1) > 0$

$x \in \left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$

Question7

Let $f : (0, 1) \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{1 - e^{-x}}$, and $g(x) = (f(-x) - f(x))$. Consider two statements
(I) g is an increasing function in (0, 1)
(II) g is one-one in (0, 1)
Then,
[25-Jan-2023 Shift 1]

Options:

A. Only (I) is true

B. Only (II) is true

C. Neither (I) nor (II) is true

D. Both (I) and (II) are true

Answer: D

Solution:

Solution:

$g(x) = f(-x) - f(x) = \frac{1 + e^x}{1 - e^x}$

$2e^x$

$\Rightarrow g$ is increasing in $(0, 1)$
$\Rightarrow g$ is one-one in $(0, 1)$

Question8

Let $x = 2$ be a local minima of the function $f(x) = 2x^4 - 18x^2 + 8x + 12, x \in (-4, 4)$. If M is local maximum value of the function f in $(-4, 4)$, then M =
[25-Jan-2023 Shift 1]

Options:

A. $12\sqrt{6} - \frac{33}{2}$

B. $12\sqrt{6} - \frac{31}{2}$

C. $18\sqrt{6} - \frac{33}{2}$

D. $18\sqrt{6} - \frac{31}{2}$

Answer: A

Solution:

Solution:
$f'(x) = 8x^3 - 36x + 8 = 4(2x^3 - 9x + 2)$
$f'(x) = 0$
$\therefore x = \frac{\sqrt{6} - 2}{2}$
Now
$f(x) = \left(x^2 - 2x - \frac{9}{2}\right)(2x^2 + 4x - 1) + 24x + 7.5$
$\therefore f\left(\frac{\sqrt{6} - 2}{2}\right) = M = 12\sqrt{6} - \frac{33}{2}$

Question9

Let the function $f(x) = 2x^3 + (2p - 7)x^2 + 3(2p - 9)x - 6$ have a maxima for some value of $x < 0$ and a minima for some value of $x > 0$. Then, the set of all values of p is
[25-Jan-2023 Shift 2]

Options:

A. $\left(\frac{9}{2}, \infty\right)$

B. $\left(0, \frac{9}{2}\right)$

D. $\left(-\frac{9}{2}, \frac{9}{2}\right)$

Answer: C

Solution:

Solution:

$f(x) = 2x^3 + (2p-7)x^2 + 3(2p-9)x - 6$

$f'(x) = 6x^2 + 2(2p-7)x + 3(2p-9)$

$f'(0) < 0$

$\therefore 3(2p-9) < 0$

$p < \frac{9}{2}$

$p \in \left(-\infty, \frac{9}{2}\right)$

Question10

If the equation of the normal to the curve $y = \frac{x-a}{(x+b)(x-2)}$ at the point $(1, -3)$ is $x - 4y = 13$ then the value of $a + b$ is equal to _______. [29-Jan-2023 Shift 2]

Answer: 4

Solution:

Solution:

$y = \frac{x-a}{(x+b)(x-2)}$

At point $(1, -3)$,

$-3 = \frac{1-9}{(1+b)(1-2)}$

$\Rightarrow 1 - a = 3(1+b) \ldots (1)$

Now, $y = \frac{x-a}{(x+b)(x-2)}$

$\Rightarrow \frac{dy}{dx} = \frac{(x+b)(x-2) \times (1) - (x-a)(2x+b-2)}{(x+b)^2(x-2)^2}$

At $(1, -3)$ slope of normal is $\frac{1}{4}$ hence $\frac{dy}{dx} = -4$, So, $-4 = \frac{(1+b)(-1) - (1-a)b}{(1+b)^2(-1)^2}$

Using equation (1)

$\Rightarrow -4 = \frac{(1+b)(-1) - 3(b+1)b}{(1+b)^2}$

$\Rightarrow -4 = \frac{(-1) - 3b}{(1+b)}(b \neq -1)$

$\Rightarrow b = -3$

So, $a = 7$

Hence, $a + b = 7 - 3 = 4$

Question11

Let $\alpha_1, \alpha_2, \ldots, \alpha_7$ be the roots of the equation $x^7 + 3x^5 - 13x^3 - 15x = 0$ and $|\alpha_1| \geq |\alpha_2| \geq \ldots \geq |\alpha_7|$.
Then $\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$ is equal to ______.
[29-Jan-2023 Shift 2]

Answer: 9

Solution:

Solution:
Given equation can be rearranged as
$x(x^6 + 3x^4 - 13x^2 - 15) = 0$
clearly $x = 0$ is one of the root and other part can be observed by replacing $x^2 = t$ from which we have
$t^3 + 3t^2 - 13t - 15 = 0$
$\Rightarrow (t-3)(t^2 + 6t + 5) = 0$
So, $t = 3, t = -1, t = -5$
Now we are getting $x^2 = 3, x^2 = -1, x^2 = -5$
$\Rightarrow x = \pm\sqrt{3}, x = \pm i, x = \pm\sqrt{5}i$
From the given condition $|\alpha_1| \geq |\alpha_2| \geq \ldots . \geq |\alpha_7|$
We can clearly say that $|\alpha_7| = 0$ and
and$|\alpha_6| = \sqrt{5} = |\alpha_5|$
and $|\alpha_4| = \sqrt{3} = |\alpha_3|$ and $|\alpha_2| = 1 = |\alpha_1|$
So we can have, $\alpha_1 = \sqrt{5}i, \alpha_2 = -\sqrt{5}i, \alpha_3 = \sqrt{3}i,$
$\alpha_4 = -\sqrt{3}, \alpha_5 = i, \alpha_6 = -i$
Hence
$\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$
$= 1 - (-3) + 5 = 9$

Question12

The number of points on the curve $y = 54x^5 - 135x^4 - 70x^3 + 180x^2 + 210x$ at which the normal lines are parallel to $x + 90y + 2 = 0$ is :
[30-Jan-2023 Shift 1]

Options:

A. 2

B. 3

C. 4

D. 0

Answer: C

Solution:

Normal of line is parallel to line x + 90y + 2 = 0

$m_N = -\frac{1}{90}$

$-\left(\frac{dx}{dy}\right)_{(x_1y_1)} = -\frac{1}{90} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1y_1)} = 90$

Now,

$\frac{dy}{dx} = 270x^4 - 540x^3 - 210x^2 + 360x + 210 = 90$

$\Rightarrow x = 1, 2, \frac{-2}{3}, \frac{-1}{3}$

(4) normals

Question13

If the functions $f(x) = \frac{x^3}{3} + 2b\,bx + \frac{ax^2}{2}$ and $g(x) = \frac{x^3}{3} + ax + bx^2$, $a \neq 2b$ have a common extreme point, then a + 2b + 7 is equal to [30-Jan-2023 Shift 2]

Options:

A. 4

B. $\frac{3}{2}$

C. 3

D. 6

Answer: D

Solution:

Solution:

$f'(x) = x^2 + 2b + ax$

$g'(x) = x^2 + a + 2bx$

$(2b - a) - x(2b - a) = 0$

$\therefore x = 1$ is the common root

Put $x = 1$ in $f'(x) = 0$ or $g'(x) = 0$

$1 + 2b + a = 0$

$7 + 2b + a = 6$

Question14

A wire of length 20m is to be cut into two pieces. A piece of length ℓ_1 is bent to make a square of area A_1 and the other piece of length ℓ_2 is made into a circle of area A_2. If $2A_1 + 3A_2$ is minimum then $(\pi\ell_1) : \ell_2$ is equal to: [31-Jan-2023 Shift 1]

Options:

A. 6 : 1

C. 1 : 6

D. 4 : 1

Answer: A

Solution:

Solution:

$\ell_1 + \ell_2 = 20 \Rightarrow \frac{d\ell_2}{d\ell_1} = -1$

$A_1 = \left(\frac{\ell_1}{4}\right)^2$ and $A_2 = \pi\left(\frac{\ell_2}{2\pi}\right)^2$

Let $S = 2A_1 + 3A_2 = \frac{\ell_1^{\,2}}{8} + \frac{3\ell_2^{\,2}}{4\pi}$

$\Rightarrow \frac{\ell_1}{4} = \frac{6\ell_2}{4\pi} \Rightarrow \frac{\pi\ell_1}{\ell_2} = 6$

Question15

Let $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$, $x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$. If α a β respectively are the maximum and the minimum values of f, then [1-Feb-2023 Shift 1]

Options:

A. $\beta^2 - 2\sqrt{\alpha} = \frac{19}{4}$

B. $\beta^2 + 2\sqrt{\alpha} = \frac{19}{4}$

C. $\alpha^2 - \beta^2 = 4\sqrt{3}$

D. $\alpha^2 + \beta^2 = \frac{9}{2}$

Answer: A

Solution:

Solution:

$C_1 \to C_1 + C_2 + C_3$

$$f(x) = \begin{vmatrix} 2+\sin 2x & \cos^2 x & \sin 2x \\ 2+\sin 2x & 1+\cos^2 x & \sin 2x \\ 2+\sin 2x & \cos^2 x & 1+\sin 2x \end{vmatrix}$$

$$f(x) = (2+\sin 2x)\begin{vmatrix} 1 & \cos^2 x & \sin 2x \\ 1 & 1+\cos^2 x & \sin 2x \end{vmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$f(x) = 2 + \sin 2x \begin{vmatrix} 1 & \cos^2 x & \sin 2x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= (2 + \sin 2x)(1) = 2 + \sin 2x$

$= \sin 2x \in \left[\frac{\sqrt{3}}{2}, 1\right]$

Hence $2 + \sin 2x \in \left[2 + \frac{\sqrt{3}}{2}, 3\right]$

Question16

Let $f(x) = 2x + \tan^{-1}x$ and $g(x) = \log_e\left(\sqrt{1 + x^2} + x\right)$, $x \in [0, 3]$. Then [1-Feb-2023 Shift 1]

Options:

A. There exists $x \in [0, 3]$ such that $f'(x) < g'(x)$

B. $\max f(x) > \max g(x)$

C. There exist $0 < x_1 < x_2 < 3$ such that $f(x) < g(x)$, $\forall x \in (x_1, x_2)$

D. $\min f'(x) = 1 + \max g'(x)$

Answer: B

Solution:

Solution:

$f(x) = 2x + \tan^{-1}x$ and $g(x) = \ln\left(\sqrt{1 + x^2} + x\right)$ and $x \in [0, 3]$

$g'(x) = \frac{1}{\sqrt{1 + x^2}}$

Now, $0 \le x \le 3$
$0 \le x^2 \le 9$
$1 \le 1 + x^2 \le 10$

So, $2 + \frac{1}{10} \le f'(x) \le 3$

$\frac{21}{10} \le f'(x) \le 3$ and $\frac{1}{\sqrt{10}} \le g'(x) \le 1$

option (4) is incorrect
From above, $g'(x) < f'(x)$ $\forall x \in [0, 3]$
Option (1) is incorrect.
$f'(x)$&$g'(x)$ both positive so $f(x)$&$g(x)$ both are increasing
So, max(f(x). at $x = 3$ is $6 + \tan^{-1}3$
Max(g(x) at $x = 3$ is $\ln(3 + \sqrt{10})$
And $6 + \tan^{-1}3 > \ln(3 + \sqrt{10})$
Option (2) is correct

Question17

If $f(x) = x^2 + g'(1)x + g''(2)$ and

then the value of $f(4) - g(4)$ is equal to ______.
[1-Feb-2023 Shift 1]

Answer: 14

Solution:

Solution:

$f(x) = x^2 + g'(1)x + g''(2)$
$f'(x) = 2x + g'(1)$
$f''(x) = 2$
$g(x) = f(1)x^2 + x[2x + g'(1)] + 2$
$g'(x) = 2f(1)x + 4x + g'(1)$
$g''(x) = 2f(1) + 4$
$g''(x) = 0$
$2f(1) + 4 = 0$
$f(1) = -2$
$-2 = 1 + g'(1) = g'(1) = -3$
So, $f'(x) = 2x - 3$
$f(x) = x^2 - 3x + c$
$c = 0$
$f(x) = x^2 - 3x$
$g(x) = -3x + 2$
$f(4) - g(4) = 14$

Question18

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that $f'(x) + f(x) = \int_0^2 f(t)\,dt$. If $f(0) = e^{-2}$, then $2f(0) - f(2)$ is equal to ______.
[1-Feb-2023 Shift 1]

Answer: 1

Solution:

$\frac{dy}{dx} + y = k$
$y \cdot e^x = k \cdot e^x + c$
$f(0) = e^{-2}$
$\Rightarrow \quad c = e^{-2} - k$
$\therefore \quad y = k + (e^{-2} - k)e^{-x}$
$\Rightarrow \quad k = e^{-2} - 1$
$\therefore \quad y = (e^{-2} - 1) + e^{-x}$
$f(2) = 2e^{-2} - 1,\ f(0) = e^{-2}$
$2f(0) - f(2) = 1$

Question19

The sum of the abosolute maximum and minimum values of the function $f(x) = |x^2 - 5x + 6| - 3x + 2$ in the interval $[-1, 3]$ is equal to :
[1-Feb-2023 Shift 2]

Options:

A. 10

B. 12

C. 13

D. 24

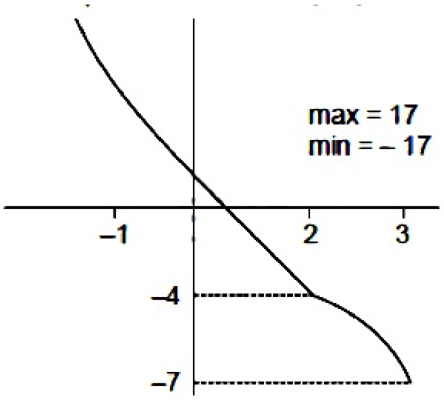
Answer: A

Solution:

Solution:

$f(x) = |x^2 - 5x + 6| - 3x + 2$

$$f(x) = \begin{cases} x^2 - 8x + 8 & ; x \in [-1, 2] \\ -x^2 + 2x - 4 & ; x \in [2, 3] \end{cases}$$



Question20

Let the tangents to the curve $x^2 + 2x - 4y + 9 = 0$ at the point P(1, 3) on it meet the y-axis at A. Let the line passing through P and parallel to the line $x - 3y = 6$ meet the parabola $y^2 = 4x$ at B. If B lies on the line $2x - 3y = 8$, then $(AB)^2$ is equal to ______.
[6-Apr-2023 shift 1]

Answer: 292

Solution:

$C : x^2 + 2x - 4y + 9 = 0$
$C : (x+1)^2 = 4(y-2)$
$T_{P(1,3)} : x \cdot 1 + (x+1) - 2(y+3) + 9 = 0$
$: 2x - 2y + 4 = 0$
$T_p : x - y + 2 = 0$
$A : (0, 2)$
Line $|$ to $x - 3y = 6$ passes $(1, 3)$ is $x - 3y + 8 = 0$
Meet parabola $: y^2 = 4x$
$\Rightarrow y^2 = 4(3y - 8)$
$\Rightarrow y^2 - 12y + 32 = 0$
$\Rightarrow (y-8)(y-4) = 0$
$\Rightarrow$ point of intersection are
$(4, 4)\&(16, 8)$ lies on $2x - 3y = 8$
Hence $A : (0, 2)$
$B : (16, 8)$
$(AB)^2 = 256 + 36 = 292$

Question21

Let a curve $y = f(x), x \in (0, \infty)$ pass through the points $P\left(1, \frac{3}{2}\right)$ and $Q\left(a, \frac{1}{2}\right)$. If the tangent at any point $R(b, f(b))$ to the given curve cuts the y-axis at the points $S(0, c)$ such that $bc = 3$, then $(PQ)^2$ is equal to ______ :
[6-Apr-2023 shift 2]

Answer: 5

Solution:

Equation of tangent at $R(b, f(2))$ is
$y - f(b) = f'(b) \cdot (x - b)$
which passes through $(0, c)$
$\Rightarrow c - f(b) = f'(b) \cdot (-b)$
$\Rightarrow \frac{3}{b} - f(b) = f(b) \cdot (-b)$
$\Rightarrow \frac{bf'(b) - f(b)}{b^2} = -\frac{3}{b^3}$
$\Rightarrow d\left(\frac{f(b)}{b}\right) = -\frac{3}{b^3} \Rightarrow \frac{f(b)}{b} = \frac{3}{2b^2} + \lambda$
Which passes through (1, 3 / 2)
$\Rightarrow \frac{3}{2} = \frac{3}{2} + \lambda \Rightarrow \lambda = 0$
$\Rightarrow f(b) = \frac{3}{2b}$
$f(a) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{3}{2b} \Rightarrow b = 3$

$\Rightarrow c = 1 \Rightarrow Q(3, 1/2)$
$\Rightarrow PQ^2 = 2^2 + (1)^2 = 5$

Question22

The number of points, where the curve $y = x^5 - 20x^3 + 50x + 2$ crosses the x-axis is ______ :
[6-Apr-2023 shift 2]

Answer: 5

Solution:

Solution:

$y = x^5 - 20x^3 + 50x + 2$

$\frac{dy}{dx} = 5x^4 - 60x^2 + 50 = 5(x^4 - 12x^2 + 10)$

$\frac{dy}{dx} = 0 \Rightarrow x^4 - 12x^2 + 10 = 0$

$\Rightarrow x^2 = \frac{12 \pm \sqrt{144 - 40}}{2}$

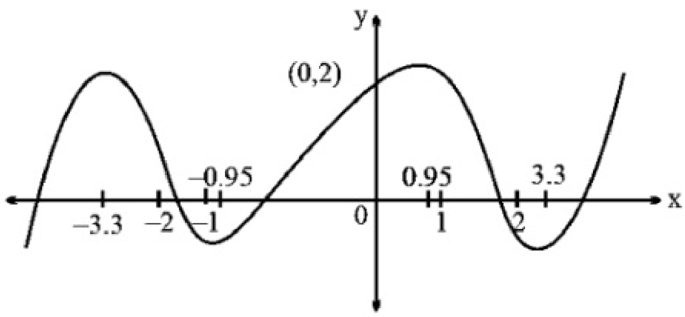
$\Rightarrow x^2 = 6 \pm \sqrt{26} \Rightarrow x^2 \approx 6 \pm 5.1$

$\Rightarrow x^2 \approx 11.1, 0.9$

$\Rightarrow x \approx \pm 3.3, \pm 0.95$

$f(0) = 2, f(1) = +ve, f(2) = -ve$

$f(-1) = -ve, f(-2) = +ve$



The number of points where the curve cuts the x-axis $= 5$.

Question23

A square piece of tin of side 30 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. If the volume of the box is maximum, then its surface area (in cm^2) is equal to :
[10-Apr-2023 shift 1]

Options:

A. 800

B. 1025

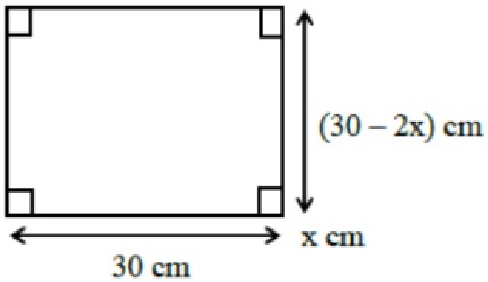
C. 900

D. 675

Answer: A

Solution:

Solution:
Let the side of the square to be cut off be x cm.
Then, the length and breadth of the box will be $(30-2x)$ cm each and the height of the box is x cm therefore,



The volume V (x) of the box is given by
$V(x) = x(30-2x)^2$
$\frac{dv}{dx} = (30-2x)^2 + 2x \times (30-2x)(-2)$
$0 = (30-2x)^2 - 4x(30-2x)$
$0 = (30-2x)[(30-2x)-4x]$
$0 = (30-2x)(30-6x)$
$x = 15, 5$
$x \neq 15$ (Not possible)
$\{\because V = 0\}$
(Not possible)
Surface area without top of the box $= \ell b + 2(bh + h\ell)$
$= (30-2x)(30-2x) + 2[(30-2x)x + (30-2x)x]$
$= [(30-2x)^2 + 4\{(30-2x)x\}.$
$= [(30-10)^2 + 4(5)(30-10)]$
$= 400 + 400$
$= 800 cm^2$

Question24

Let $g(x) = f(x) + f(1-x)$ and $f^n(x) > 0$, $x \in (0, 1)$. If g is decreasing in the interval $(0, \alpha)$ and increasing in the interval $(\alpha, 1)$, then $\tan^{-1}(2\alpha) + \tan^{-1}\left(\frac{\alpha+1}{\alpha}\right)$ is equal to
[10-Apr-2023 shift 2]

Options:

A. $\frac{5\pi}{4}$

B. π

C. $\frac{3\pi}{4}$

D. $\frac{3\pi}{2}$

Answer: B

Solution:

$g(x) = f(x) + f(1 - x) \& f''(x) > 0, x \in (0, 1)$

$g'(x) = f'(x) - f'(1 - x) = 0$

$\Rightarrow f'(x) = f'(1 - x)$

$x = 1 - x$

$x = \frac{1}{2}$

$g'(x) = 0$

at $x = \frac{1}{2}$

$g''(x) = f''(x) + f''(1 - x) > 0$

g is concave up

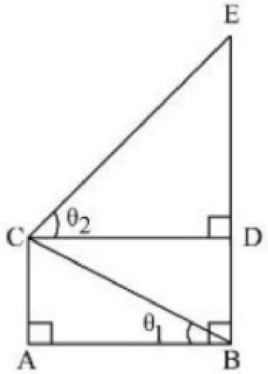
hence $\alpha = \frac{1}{2}$

$\tan^{-1}2\alpha + \tan^{-1}\frac{1}{\alpha} + \tan^{-1}\frac{\alpha + 1}{\alpha}$

$\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

Question25

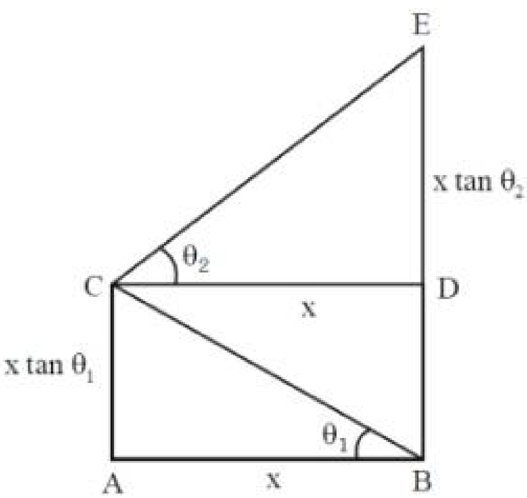
In the figure, $\theta_1 + \theta_2 = \frac{\pi}{2}$ and $\sqrt{3}(BE) = 4(AB)$. If the area of $\triangle CAB$ is $2\sqrt{3} - 3$ unit ^{2}, when $\frac{\theta_2}{\theta_1}$ is the largest, then the perimeter (in unit) of $\triangle CED$ is equal to ______.



[10-Apr-2023 shift 2]

Answer: 6

Solution:



$\sqrt{3}\,BE = 4\,AB$

$Ar(\triangle CAB) = 2\sqrt{3} - 3$
$\frac{1}{2}x^2 \tan\theta_1 = 2\sqrt{3} - 3$
$BE = BD + DE$
$= x(\tan\theta_1 + \tan\theta_2)$
$BE = AB(\tan\theta_1 + \cot\theta_1)$
$\frac{4}{\sqrt{3}} \tan\theta_1 + \cot\theta_1 \Rightarrow \tan\theta_1 = \sqrt{3}, \frac{1}{\sqrt{3}}$
$\theta_1 = \frac{\pi}{6}\ \theta_2 = \frac{\pi}{3}$
$\theta_1 = \frac{\pi}{3}\ \theta_2 = \frac{\pi}{6}$
as $\frac{\theta_2}{\theta_1}$ is largest $\therefore \theta_1 = \frac{\pi}{6}\theta_2 = \frac{\pi}{3}$
$\therefore x^2 = \frac{(2\sqrt{3} - 3) \times 2}{\tan\theta_1} = \frac{\sqrt{3}(2 - \sqrt{3}) \times 2}{\tan\frac{\pi}{6}}$
$x^2 = 12 - 6\sqrt{3} = (3 - \sqrt{3})^2$
$x = 3 - \sqrt{3}$
Perimeter of $\triangle CED$
$= CD + DE + CE$
$= 3\sqrt{3} + (3 - \sqrt{3})\sqrt{3} + (3 - \sqrt{3}) \times 2 = 6$
Ans. (6)

Question26

Let the quadratic curve passing through the point (−1, 0) and touching the line y = x at (1, 1) be y = f (x). Then the x-intercept of the normal to the curve at the point (α, α + 1) in the first quadrant is ______.
[10-Apr-2023 shift 2]

Answer: 11

Solution:

Solution:
$f(x) = (x + 1)(ax + b)$
$1 = 2a + 2b$
$f'(x) = (ax + b) + a(x + 1)$
$1 = (3a + b)$
$\Rightarrow b = 1/4, a = 1/4$
$f(x) = \frac{(x + 1)^2}{4}$
$f'(x) = \frac{x}{2} + \frac{1}{2}\ \alpha + 1 = \frac{(\alpha + 1)^2}{4}, \alpha > -1$
$\alpha + 1 = 4$
$\alpha = 3$
normal at (3, 4)
$y - 4 = -\frac{1}{2}(x - 3)$
$y = 0\ x = 8 + 3$
Ans. 11

Question27

Let f : [2, 4] → ℝ be a differentiable function such that $(x\log_e x)f'(x) + (\log_e x)f(x) + f(x) \geq 1, x \in [2, 4]$ with $f(2) = \frac{1}{2}$ and $f(4) = \frac{1}{4}$.
Consider the following two statements :
(A) : $f(x) \leq 1$, for all $x \in [2, 4]$
(B) : $f(x) \geq \frac{1}{8}$, for all $x \in [2, 4]$
Then,
[11-Apr-2023 shift 1]

Options:

A. Only statement (B) is true

B. Only statement (A) is true

C. Neither statement (A) nor statement (B) is true

D. Both the statements (A) and (B) are true

Answer: D

Solution:

Solution:

$x\ell nxf f'(x) + \ln xf(x) + f(x) \geq 1, x \in [2, 4]$

And $f(2) = \frac{1}{2}, f(4) = \frac{1}{4}$

Now $x\ln x, \frac{dy}{dx} + (\ln + 1)y \geq 1$

$\frac{d}{dx}(y \cdot x\ln x) \geq 1$

$\frac{d}{dx}(f(x) \cdot x\ln x) \geq 1$

$\Rightarrow \frac{d}{dx}(x\ln x f(x) - x) \geq 0, x \in [2, 4]$

$\Rightarrow$ The function $g(x) = x\ln x f(x) - x$ is increasing in [2, 4]

And $g(2) = 2\ln 2 f(2) - 2 = \ln 2 - 2$

$g(2) = 4\ln 4 f(4) - 4 = \ln 4 - 4$

$= 2(\ln 2 - 2)$

Now $g(2) \leq g(x) \leq g(4)$

$\text{Ln } 2 - 2 \leq x\ln x f(x) - x \leq 2(\ln 2 - 2)$

$\frac{\ln 2 - 2}{x\ln x} + \frac{1}{\ln x} \leq f(x) \leq \frac{2(\ln 2 - 2)}{x\ln x} + \frac{1}{\ln x}$

Now for $x \in [2, 4]$

$\frac{2(\ln 2 - 2)}{x\ln x} + \frac{1}{\ln x} < \frac{2(\ln 2 - 2)}{2\ln 2} + \frac{1}{\ln 2} = 1 - \frac{1}{\ln 2} < 1$

$\Rightarrow f(x) \leq 1$ for $x \in [2, 4]$

$\frac{\ln 2 - 2}{x\ln x} + \frac{1}{\ln x} \geq \frac{\ln 2 - 2}{4\ln 4} + \frac{1}{\ln 4} = \frac{1}{8} + \frac{1}{2\ln 2} > \frac{1}{8}$

$\Leftrightarrow f(x) \geq \frac{1}{8}$ for $x \in [2, 4]$ Hence both A and B are true.

Question28

If the local maximum value of the function $f(x) = \left(\frac{\sqrt{3e}}{2\sin x}\right)^{\sin^2 x}, x \in \left(0, \frac{\pi}{2}\right)$, is $\frac{k}{e}$, then $\left(\frac{k}{e}\right)^8 + \frac{k^8}{e^5} + k^8$ is equal to

Options:

A. $e^3 + e^6 + e^{11}$

B. $e^5 + e^6 + e^{11}$

C. $e^3 + e^5 + e^{11}$

D. $e^3 + e^6 + e^{10}$

Answer: A

Solution:

Solution:

$$\text{Let } y = \left(\frac{\sqrt{3}e}{2\sin x}\right)^{\sin^2 x}$$

$$\ln y = \sin^2 x \cdot \ln\left(\frac{\sqrt{3}e}{2\sin x}\right)$$

$$\frac{1}{y}y' = \ln\left(\frac{\sqrt{3}e}{2\sin x}\right) 2\sin x \cos x + \sin^2 x \frac{2\sin x}{\sqrt{3}e} \frac{\sqrt{3}e}{2}(-\operatorname{cosec} x \cot x)$$

$$\Rightarrow \sin x \cos\left[2\ln\left(\frac{\sqrt{3}e}{2\sin x}\right) - 1\right] = 0$$

$$\Rightarrow \ln\left(\frac{3e}{4\sin^2 x}\right) = 1 \Rightarrow \frac{3e}{4\sin^2 x} = e \Rightarrow \sin^2 x = \frac{3}{4}$$

$$\Rightarrow \sin x = \frac{\sqrt{3}}{2}\left(\text{ as } x \in \left(0, \frac{\pi}{2}\right)\right)$$

$$\Rightarrow \text{local max value } = \left(\frac{\sqrt{3}e}{\sqrt{3}}\right)^{3/4} = e^{3/8} = \frac{k}{e}$$

$$\Rightarrow k^8 = e^{11}$$

$$\Rightarrow \left(\frac{k}{e}\right)^8 + \frac{k^8}{e^5} + k^8 = e^3 + e^6 + e^{11}$$

Question29

$$\max_{0 \le x \le \pi}\left\{x - 2\sin x\cos x + \frac{1}{3}\sin 3x\right\} =$$

[13-Apr-2023 shift 1]

Options:

A. 0

B. π

C. $\frac{5\pi + 2 + 3\sqrt{3}}{6}$

D. $\frac{\pi + 2 - 3\sqrt{3}}{6}$

Answer: C

Solution:

$f(x) = x - \sin 2x + \frac{1}{3}\sin 3x$

$f'(x) = 1 - 2\cos 2x + \cos 3x = 0$

$x = \frac{5\pi}{6}, \frac{\pi}{6}$

$\therefore f''(x) = 4\sin 2x - 3\sin 3x$

$f''\left(\frac{5\pi}{6}\right) < 0$

$\Rightarrow \left(\frac{5\pi}{6}\right)$ is point of maxima

$f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2} + \frac{1}{3}$

Question30

The set of all $a \in R$ for which the equation $x|x-1| + |x+2| + a = 0$ has exactly one real root, is
[13-Apr-2023 shift 1]

Options:

A. $(-\infty, -3)$

B. $(-\infty, \infty)$

C. $(-6, \infty)$

D. $(-6, -3)$

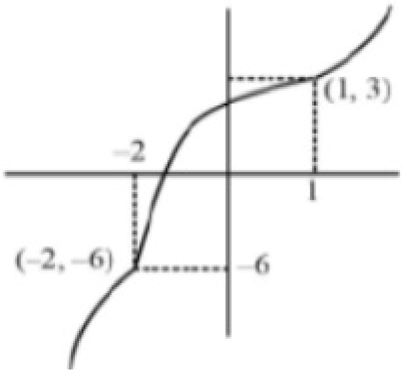
Answer: B

Solution:

$f(x) = x|x-1| + |x+2|$

$x|x-1| + |x+2| + a = 0$

$x|x-1| + |x+2| = -a$



All values are increasing.

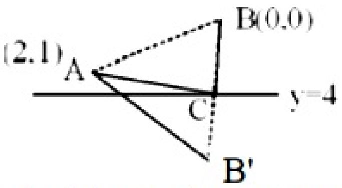
Question31

Consider the triangles with vertices A(2, 1), B(0, 0) and C(t, 4), $t \in [0, 4]$. If the maximum and the minimum perimeters of such triangles are obtained at $t = \alpha$ and $t = \beta$ respectively, then $6\alpha + 21\beta$ is equal to ________
[15-Apr-2023 shift 1]

Answer: 48

Solution:

A(2, 1), B(0, 0), C(t, 4) : t ∈ [0, 4]



B1(0, 8)≡ image of B w.r.t. y = 4

for AC + BC + AB to be minimum

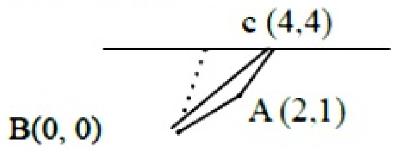
$m_{AB'} = \frac{-7}{2}$

line $AB_1 = 7x + 2y = 16$

$C\left(\frac{8}{7}, 4\right)$

$\beta = \frac{8}{7}$

For max. perimeter



$\alpha = 4$

$AB = \sqrt{5} : BC = 4\sqrt{2}, AC = \sqrt{13}$

$6\alpha + 21\beta = 24 + 24 = 48$

Question32

The number of distinct real roots of the equation $x^7 - 7x - 2 = 0$ is
[24-Jun-2022-Shift-2]

Options:

A. 5

B. 7

C. 1

D. 3

Answer: D

Solution:

Given equation $x^7 - 7x - 2 = 0$

Let $f(x) = x^7 - 7x - 2$

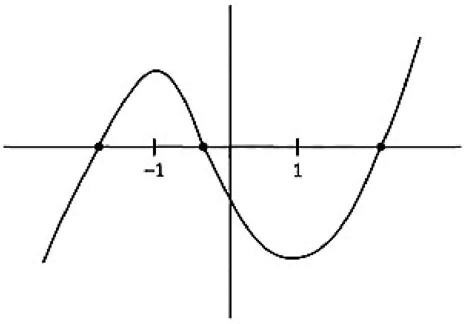
$f'(x) = 7x^6 - 7 = 7(x^6 - 1)$

and $f'(x) = 0 \Rightarrow x = +1$

and $f(-1) = -1 + 7 - 2 = 5 > 0$

$f(1) = 1 - 7 - 2 = -8 < 0$

So, roughly sketch of $f(x)$ will be



So, number of real roots of $f(x) = 0$ and 3

Question33

Let λ* be the largest value of λ for which the function $f_\lambda(x) = 4\lambda x^3 - 36\lambda x^2 + 36x + 48$ is increasing for all $x \in R$. Then $f_{\lambda*}(1) + f_{\lambda*}(-1)$ is equal to :
[24-Jun-2022-Shift-2]

Options:

A. 36

B. 48

C. 64

D. 72

Answer: D

Solution:

$\because f_\lambda(x) = 4\lambda x^3 - 36\lambda x^2 + 36\lambda + 48$

$\therefore f_\lambda'(x) = 12(\lambda x^2 - 6\lambda x + 3)$

For $f_\lambda(x)$ increasing : $(6\lambda)^2 - 12\lambda \le 0$

$\therefore \lambda \in \left[0, \frac{1}{3}\right]$

$\therefore \lambda * = \frac{1}{3}$

Now, $f_\lambda * (x) = \frac{4}{3}x^3 - 12x^2 + 36x + 48$

$\therefore f_\lambda * (1) + f_\lambda * (-1) = 73\frac{1}{2} - 1\frac{1}{2} = 72$

Question34

Let f(x) be a polynomial function such that $f(x) + f'(x) + f''(x) = x^5 + 64$. Then, the value of $\lim_{x \to 1} \frac{f(x)}{x-1}$ is equal to:

[25-Jun-2022-Shift-1]

Options:

A. -15

B. -60

C. 60

D. 15

Answer: A

Solution:

$$\lim_{x \to 1} \frac{f(x)}{x-1}$$

$$f(x) + f'(x) + f''(x) = x^5 + 64$$

Let $f(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$

$$f'(x) = 5x^4 + 4ax^3 + 3bx^2 + 2cx + d$$

$$f''(x) = 20x^3 + 12ax^2 + 6bx + 2c$$

$$x^5(a+5)x^4 + (b+4a+20)x^3 + (c+3b+12a)x^2 + (d+2c+6b)x + e + d + 2c = x^5 + 64$$

$$\Rightarrow a + 5 = 0$$

$$b + 4a + 20 = 0$$

$$c + 3b + 12a = 0$$

$$d + 2c + 6b = 0$$

$$e + d + 2c = 64$$

$$\therefore a = -5,\ b = 0,\ c = 60,\ d = -120,\ e = 64$$

$$\therefore f(x) = x^5 - 5x^4 + 60x^2 - 120x + 64$$

Now, $\lim_{x \to 1} \frac{x^5 - 5x^4 + 60x^2 - 120x + 64}{x-1}$ is ($\frac{0}{0}$ form)

By L' Hospital rule

$$\lim_{x \to 1} \frac{5x^4 - 20x^3 + 120x - 120}{1}$$

$$= -15$$

Question35

Let f : R → R and g : R → R be two functions defined by $f(x) = \log_e(x^2 + 1) - e^{-x} + 1$ and $g(x) = \frac{1 - 2e^{2x}}{e^x}$. Then, for which of the following range of α, the inequality $f\left(g\left(\frac{(\alpha - 1)^2}{3}\right)\right) > f\left(g\left(\alpha - \frac{5}{3}\right)\right)$ holds ?

[25-Jun-2022-Shift-1]

Options:

A. (2, 3)

B. (−2, −1)

C. (1, 2)

D. (−1, 1)

Answer: A

Solution:

$f(x) = \log_e(x^2 + 1) - e^{-x} + 1$

$f'(x) = \dfrac{2x}{x^2 + 1} + e^{-x}$

$= \dfrac{2}{x + \dfrac{1}{x}} + e^{-x} > 0 \ \forall x \in R$

$g(x) = e^{-x} - 2e^x$

$g'(x) - -e^{-x} - 2e^x < 0 \quad \forall x \in R$

⇒f(x) is increasing and g (x) is decreasing function.

$f\left(g\left(\dfrac{(\alpha - 1)^2}{3}\right)\right) > f\left(g\left(\alpha - \dfrac{5}{3}\right)\right)$

$\Rightarrow \dfrac{(\alpha - 1)^2}{3} < \alpha - \dfrac{5}{3}$

$= \alpha^2 - 5\alpha + 6 < 0$

$= (\alpha - 2)(\alpha - 3) < 0$

$= \alpha \in (2, 3)$

Question36

Water is being filled at the rate of $1 cm^3 / sec$ in a right circular conical vessel (vertex downwards) of height 35 cm and diameter 14 cm. When the height of the water level is 10 cm, the rate (in cm^2 / sec) at which the wet conical surface area of the vessel increases is
[25-Jun-2022-Shift-2]

Options:

A. 5

B. $\frac{\sqrt{21}}{5}$

C. $\frac{\sqrt{26}}{5}$

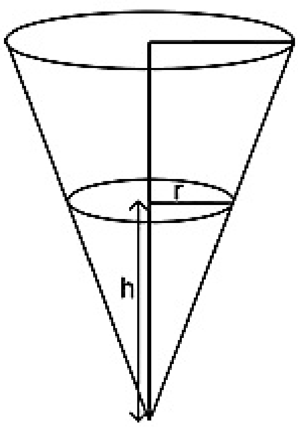
D. $\frac{\sqrt{26}}{10}$

Answer: C

Solution:

$\because V = \frac{1}{3}\pi r^2 h$ and $\frac{r}{h} = \frac{7}{35} = \frac{1}{5}$

$\Rightarrow V = \frac{1}{75}\pi h^3$



$\frac{dV}{dt} = \frac{1}{25}\pi h^2 \frac{dh}{dt} = 1$

$\Rightarrow \frac{dh}{dt} = \frac{25}{\pi h^2}$

Now, $S = \pi r l = \pi\left(\frac{h}{5}\right)\sqrt{h^2 + \frac{h^2}{25}} = \frac{\pi}{25}\sqrt{26}h^2$

$\Rightarrow \frac{dS}{dt} = \frac{2\sqrt{26}\pi h}{25} \cdot \frac{dh}{dt} = \frac{2\sqrt{26}}{h}$

$\Rightarrow \frac{dS}{dt_{(h = 10)}} = \frac{\sqrt{26}}{5}$

Question37

If the angle made by the tangent at the point (x_0, y_0) on the curve $x = 12(t + \sin t \cos t)$, $y = 12(1 + \sin t)^2$, $0 < t < \frac{\pi}{2}$, with the positive x-axis is $\frac{\pi}{3}$, then y_0 is equal to:
[25-Jun-2022-Shift-2]

Options:

A. $6(3 + 2\sqrt{2})$

B. $3(7 + 4\sqrt{3})$

C. 27

D. 48

Answer: C

Solution:

$$\because \frac{dy}{dx} = \frac{24(1 + \sin t)\cos t}{12(1 + \cos 2t)} = \frac{1 + \sin t}{\cos t} = \tan\left(\frac{\pi}{4} + \frac{t}{2}\right)$$

$$\because \frac{dy}{dx_{(x_0, y_0)}} = \sqrt{3} = \tan\left(\frac{\pi}{4} + \frac{t}{2}\right)$$

$$\Rightarrow t = \frac{\pi}{6}$$

$$\text{So, } y_{0\left(\text{ at } t = \frac{\pi}{6}\right)} = 12\left(1 + \sin\frac{\pi}{6}\right)^2 = 27$$

Question38

Let $f(x) = |(x-1)(x^2-2x-3)| + x - 3, x \in R$. If m and M are respectively the number of points of local minimum and local maximum of f in the interval (0, 4), then m + M is equal to____
[25-Jun-2022-Shift-2]

Answer: 3

Solution:

$f(x) = |(x-1)(x+1)(x-3)| + (x-3)$

$$f(x) = \begin{cases} (x-3)(x^2) & 3 \le x \le 4 \\ (x-3)(2-x^2) & 1 \le x < 3 \\ (x-3)(x^2) & 0 < x < 1. \end{cases}$$

$$f'(x) = \begin{cases} 3x^2 - 6x & 3 < x < 4 \\ -3x^2 + 6x + 2 & 1 < x < 3 \\ 3x^2 - 6x & 0 < x < 1. \end{cases}$$

$f'(3^+) > 0 \;\; f'(3^-) < 0 \rightarrow$ Minimum
$f'(1^+) > 0 \;\; f'(1^-) < 0 \rightarrow$ Minimum
$x \in (1, 3) f'(x) = 0$ at one point $\rightarrow$ Maximum
$x \in (3, 4) f'(x) \neq 0$
$x \in (0, 1) f'(x) \neq 0$
So, 3 points.

Question39

The sum of the absolute minimum and the absolute maximum values of the
function $f(x) = 3x - x^2 + 2| - x$ in the interval $[-1, 2]$ is :
[26-Jun-2022-Shift-1]

Options:

A. $\frac{\sqrt{17}+3}{2}$

B. $\frac{\sqrt{17}+5}{2}$

C. 5

D. $\frac{9-\sqrt{17}}{2}$

Answer: A

Solution:

$$f(x) = \begin{cases} x^2 - 4x - 2 & \forall x \in \left(-1, \frac{3-\sqrt{17}}{2}\right) \\ -x^2 + 2x + 2 & \forall x \in \left(\frac{3-\sqrt{17}}{2}, 2\right). \end{cases}$$

$f'(x)$ when $x \in \left(-1, \frac{3-\sqrt{17}}{2}\right)$

$f'(x) = 2x - 4 = 0 \Rightarrow x = 2$

$f'(x) = 2(x-2) \quad \Rightarrow f'(x)$ is always $\downarrow$

$f(2) = 2\, f(-1) = 3$

$f\left(\frac{3-\sqrt{17}}{2}\right) = \frac{\sqrt{17}-3}{2}$

$f'(x)$ when $x \in \left(\frac{3-\sqrt{17}}{2}, 2\right)$

$f'(x) = -2x + 2$

$f'(x) = -2(x-1)$

$f'(x) = 0$ when $x = 1$

$f(1) = 3$

absolute minimum value $= \frac{\sqrt{17}-3}{2}$ absolute maximum value $= 3$

$\text{Sum} = \frac{\sqrt{17}-3}{2} + 3 = \frac{\sqrt{17}+3}{2}$

Question40

Let S be the set of all the natural numbers, for which the line $\frac{x}{a} + \frac{y}{b} = 2$ is a tangent to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at the point (a, b), ab ≠ 0. Then :
[26-Jun-2022-Shift-1]

Options:

A. S = φ

B. n(S) = 1

C. $S = \{2k : k \in N\}$

D. $S = N$

Answer: D

Solution:

Solution:

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$$

Differentiating both sides with respect to x, we get

$$\Rightarrow n \cdot \left(\frac{x}{a}\right)^{n-a} \cdot \frac{1}{a} + n \cdot \left(\frac{y}{b}\right)^{n-1} \cdot \frac{1}{b} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{1}{a} \cdot \left(\frac{x}{a}\right)^{n-1}}{\frac{1}{b}\left(\frac{y}{b}\right)^{n-1}}$$

$$= -\frac{b}{a}\left(\frac{xb}{ya}\right)^{n-1}$$

Now, $\frac{dy}{dx}$ at (a, b) $= -\frac{b}{a}\left[\frac{ab}{ba}\right]^{n-1} = -\frac{b}{a}$

Equation of tangent at (a, b) is,

$$(y - b) = -\frac{b}{a}(x - a)$$

$$\Rightarrow \frac{y - b}{b} = -\frac{x - a}{a}$$

$$\Rightarrow \frac{y}{b} - 1 = -\frac{x}{a} + 1$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 2$$

$\therefore$ It is tangent for all value of n.

Question41

Let $f(x) = 2\cos^{-1}x + 4\cot^{-1}x - 3x^2 - 2x + 10$, $x \in [-1, 1]$. If [a, b] is the range of the function f, then $4a - b$ is equal to :
[26-Jun-2022-Shift-1]

Options:

A. 11

B. $11 - \pi$

C. $11 + \pi$

D. $15 - \pi$

Answer: B

Solution:

Solution:

$f(x)=2\cos^{-1}x+4\cot^{-1}x-3x^2-2x+10\ \forall x\in[-1,1]$

$\Rightarrow f'(x)=-\frac{2}{\sqrt{1-x^2}}-\frac{4}{1+x^2}-6x-2<0\ \forall x\in[-1,1]$

So $f(x)$ is decreasing function and range of $f(x)$ is $[f(1), f(-1)]$, which is $[\pi+5, 5\pi+9]$

Now $4a-b=4(\pi+5)-(5\pi+9)$

$=11-\pi$

Question42

Consider a cuboid of sides 2x, 4x and 5x and a closed hemisphere of radius r. If the sum of their surface areas is a constant k, then the ratio x : r, for which the sum of their volumes is maximum, is :
[26-Jun-2022-Shift-2]

Options:

A. 2 : 5

B. 19 : 45

C. 3 : 8

D. 19 : 15

Answer: B

Solution:

Solution:

$\because s_1+s_2=k$

$76x^2+3\pi r^2=k$

$\therefore 152x\frac{dx}{dr}+6\pi r=0$

$\therefore \frac{dx}{dr}=\frac{-6\pi r}{152x}$

Now $V=40x^3+\frac{2}{3}\pi r^3$

$\therefore \frac{dv}{dr}=120x^2\cdot\frac{dx}{dr}+2\pi r^2=0$

$\Rightarrow 120x^2\cdot\left(\frac{-6\pi}{152}\frac{r}{x}\right)+2\pi r^2=0$

$\Rightarrow 120\left(\frac{x}{r}\right)=2\pi\left(\frac{152}{6\pi}\right)$

$\Rightarrow\left(\frac{x}{r}\right)=\frac{152}{3}\frac{1}{120}=\frac{19}{45}$

Question43

$x\frac{dy}{dx} + 2y = xe^x$, $y(1) = 0$ then the local maximum value of the function $z(x) = x^2y(x) - e^x$, $x \in R$ is :
[26-Jun-2022-Shift-2]

Options:

A. $1 - e$

B. 0

C. $\frac{1}{2}$

D. $\frac{4}{e} - e$

Answer: D

Solution:

Solution:

$x\frac{dy}{dx} + 2y = xe^x, y(1) = 0$

$\frac{dy}{dx} + \frac{2}{x}y = e^x$, then $e^{\int \frac{2}{x}dx} dx = x^2$

$y \cdot x^2 = \int x^2e^x dx$

$yx^2 = x^2e^x - \int 2xe^x dx$

$= x^2e^x - 2(xe^x - e^x) + c$

$yx^2 = x^2e^x - 2xe^x + 2e^x + c$

$yx^2 = (x^2 - 2x + 2)e^x + c$

$0 = e + c \Rightarrow c = -e$

$y(x).x^2 - e^x = (x-1)^2e^x - e$

$z(x) = (x-1)^2e^x - e$

For local maximum $z'(x) = 0$

$\therefore 2(x-1)e^x + (x-1)^2e^x = 0$

$\therefore x = -1$

And local maximum value $= z(-1)$

$= \frac{4}{e} - e$

Question44

The number of distinct real roots of $x^4 - 4x + 1 = 0$ is:
[27-Jun-2022-Shift-1]

Options:

A. 4

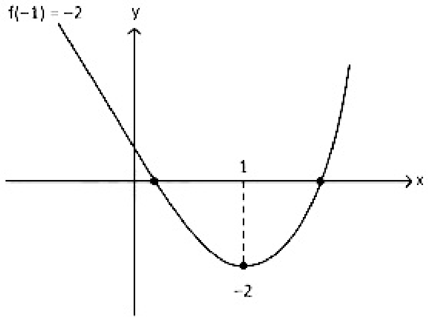
C. 1

D. 0

Answer: B

Solution:

Solution:
$f(x) = x^4 - 4x + 1 = 0$
$f'(x) = 4x^3 - 4$
$= 4(x - 1)(x^2 + 1 + x)$



Question45

The lengths of the sides of a triangle are $10 + x^2$, $10 + x^2$ and $20 - 2x^2$. If for $x = k$, the area of the triangle is maximum, then $3k^2$ is equal to: [27-Jun-2022-Shift-1]

Options:

A. 5

B. 8

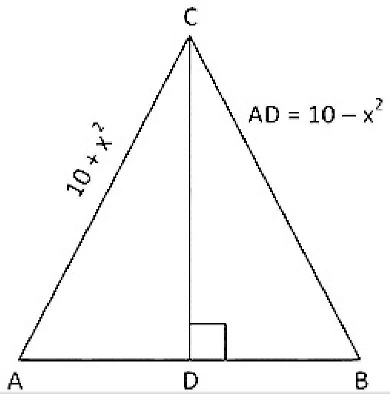
C. 10

D. 12

Answer: C

Solution:

Solution:



Area $= \frac{1}{2} \times CD \times AB = \frac{1}{2} \times 2\sqrt{10}\left|x\right|(20 - 2x^2)$
$A = \sqrt{10}\,|\,x\,|\,(10 - x^2)$
$\frac{dA}{dx} = \sqrt{10}\,\frac{|x|}{x}(10 - x^2) + \sqrt{10}\left|x\right|(-2x) = 0$
$\Rightarrow 10 - x^2 = 2x^2$
$3x^2 = 10$
$x = k$
$3k^2 = 10$

Question46

If m and n respectively are the number of local maximum and local minimum points of the function $f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$, then the ordered pair (m, n) is equal to
[27-Jun-2022-Shift-2]

Options:

A. (3, 2)

B. (2, 3)

C. (2, 2)

D. (3, 4)

Answer: B

Solution:

Solution:
$f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$
$f'(x) = 2x\left(\frac{x^4 - 5x^2 + 4}{2 + e^{x^2}}\right) = 0$
$x = 0$, or $(x^2 - 4)(x^2 - 1) = 0$
$x = 0, x = \pm 2, \pm 1$
Now, $f'(x) = \frac{2x(x + 1)(x - 1)(x + 2)(x - 2)}{\left(e^{x^2} + 2\right)}$
$f'(x)$ changes sign from positive to negative at $x = -1, 1$ So, number of local maximum points = 2
$f'(x)$ changes sign from negative to positive at $x = -2, 0, 2$ So, number of local minimum points = 3
$\therefore m = 2, n = 3$

Question47

The number of real solutions of $x^7 + 5x^3 + 3x + 1 = 0$ is equal to
[28-Jun-2022-Shift-1]

Options:

B. 1

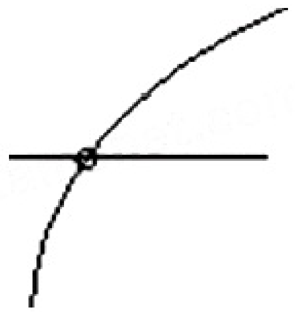
C. 3

D. 5

Answer: B

Solution:

$f(x) = x^7 + 5x^3 + 3x + 1$
$f'(x) = 7x^6 + 15x^2 + 3 > 0$
$\therefore f(x)$ is strictly increasing function



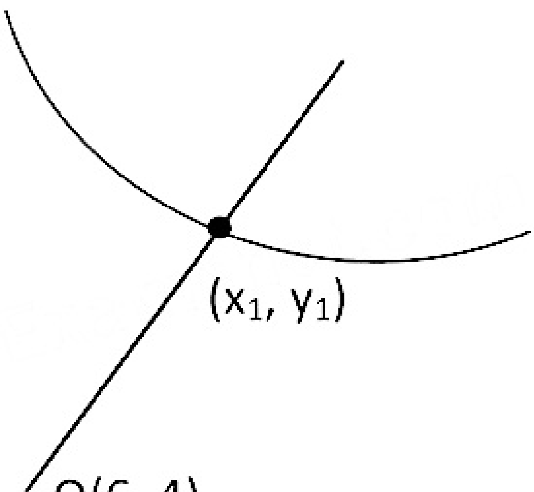
$x \to -\infty, y \to -\infty$
$x \to \infty, y \to \infty$
$\therefore$ no. of real solution $= 1$

Question48

Let I be a line which is normal to the curve $y = 2x^2 + x + 2$ at a point P on the curve. If the point Q(6, 4) lies on the line I and O is origin, then the area of the triangle OPQ is equal to___
[28-Jun-2022-Shift-1]

Answer: 13

Solution:



$$\frac{y_1 - 4}{x_1 - 6} = -\frac{1}{4x_1 + 1}$$

$\Rightarrow \frac{2x_1^2 + x_1 - 2}{x_1 - 6} = -\frac{1}{4x_1 + 1}$
$\Rightarrow 6 - x_1 = 8x_1^3 + 6x_1^2 - 7x_1 - 2$
$\Rightarrow 8x_1^3 + 6x_1^2 - 6x_1 - 8 = 0$
So $x_1 = 1 \Rightarrow y_1 = 5$

Area $= \left| \begin{matrix} 0 & 0 & 1 \\ 6 & 4 & 1 \\ 1 & 5 & 1 \end{matrix} \right| | = 13.$

Question49

Let the slope of the tangent to a curve $y = f(x)$ at (x, y) be given by $2\tan x(\cos x - y)$. If the curve passes through the point $\left(\frac{\pi}{4}, 0\right)$, then the value of $\int_0^{\pi/2} y\,dx$ is equal to:
[28-Jun-2022-Shift-2]

Options:

A. $(2 - \sqrt{2}) + \frac{\pi}{\sqrt{2}}$

B. $2 - \frac{\pi}{\sqrt{2}}$

C. $(2 + \sqrt{2}) + \frac{\pi}{\sqrt{2}}$

D. $2 + \frac{\pi}{\sqrt{2}}$

Answer: B

Solution:

$\frac{dy}{dx} = 2\tan x(\cos x - y)$
$\Rightarrow \frac{dy}{dx} + 2\tan x\, y = 2\sin x$
$I.F. = e^{\int 2\tan x\, dx} = \sec^2 x$
∴ Solution of D.E. will be
$y(x)\sec^2 x = \int 2\sin x \sec^2 x\, dx$
$y\sec^2 x = 2\sec x + c$
∵ Curve passes through $\left(\frac{\pi}{4}, 0\right)$
$\therefore c = -2\sqrt{2}$
$\therefore y = 2\cos x - 2\sqrt{2}\cos^2 x$
$\therefore \int_0^{\pi/2} y\,dx = \int_0^{\pi/2} (2\cos x - 2\sqrt{2}\cos^2 x)\,dx$
$= 2 - 2\sqrt{2} \cdot \frac{\pi}{4} = 2 - \frac{\pi}{\sqrt{2}}$

Question50

A wire of length 22m is to be cut into two pieces. One of the pieces is to be made into a square and the other into an equilateral triangle. Then, the length of the side of the equilateral triangle, so that the combined area of the square and the equilateral triangle is minimum, is :
[29-Jun-2022-Shift-1]

Options:

A. $\frac{22}{9+4\sqrt{3}}$

B. $\frac{66}{9+4\sqrt{3}}$

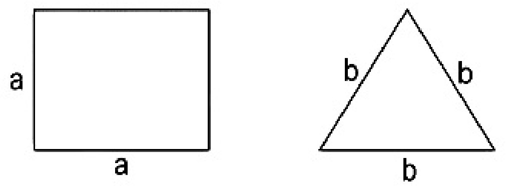
C. $\frac{22}{4+9\sqrt{3}}$

D. $\frac{66}{4+9\sqrt{3}}$

Answer: B

Solution:

Solution:



$4a + 3b = 22$

Total area $= A = a^2 + \frac{\sqrt{3}}{4}b^2$

$A = \left(\frac{22-3b}{4}\right)^2 + \frac{\sqrt{3}}{4}b^2$

$\frac{dA}{dB} = 2\left(\frac{22-3b}{4}\right)\left(\frac{-3}{4}\right) + \frac{\sqrt{3}}{4}\cdot 2b = 0$

$\Rightarrow \frac{\sqrt{3}b}{2} = \frac{3}{8}(22-3b)$

$\Rightarrow 4\sqrt{3}b = 66 - 9b$

$\therefore b = \frac{66}{9+4\sqrt{3}}$

Question51

Let $f: R \to R$ be a function defined by $f(x) = (x-3)^{n_1}(x-5)^{n_2}$, $n_1, n_2 \in N$. Then, which of the following is NOT true?
[29-Jun-2022-Shift-2]

Options:

A. For $n_1 = 3$, $n_2 = 4$, there exists $\alpha \in (3, 5)$ where f attains local maxima.

B. For $n_1 = 4$, $n_2 = 3$, there exists $\alpha \in (3, 5)$ where f attains local minima.

C. For $n_1 = 3$, $n_2 = 5$, there exists $\alpha \in (3, 5)$ where f attains local maxima.

D. For $n_1 = 4, n_2 = 6$, there exists $\alpha \in (3, 5)$ where f attains local maxima.

Answer: C

Solution:

Solution:

$$\lim_{x \to 1} \frac{(x^2 - 1)\sin^2(\pi x)}{x^4 - 2x^2 + 2x - 1}$$
$$= \lim_{x \to 1} \frac{(x^2 - 1)\sin^2(\pi x)}{(x^2 - 1)(x - 1)^2}$$
$$= \lim_{x \to 1} \frac{\sin^2(\pi x)}{(x - 1)^2}$$

Let $x = 1 + h$

$\therefore$ when $x \to 1$ then $h \to 0$

$$= \lim_{h \to 0} \frac{\sin^2(\pi(1 + h))}{(1 + h - 1)^2}$$
$$= \lim_{h \to 0} \frac{\sin^2(\pi h)}{h^2}$$
$$= \lim_{h \to 0} \pi^2 \times \frac{\sin^2(\pi h)}{(\pi h)^2}$$
$$= \pi^2 \times 1$$
$$= \pi^2$$

Question52

Let f and g be twice differentiable even functions on $(-2, 2)$ such that $f\left(\frac{1}{4}\right) = 0$, $f\left(\frac{1}{2}\right) = 0$, $f(1) = 1$ and $g\left(\frac{3}{4}\right) = 0$, $g(1) = 2$. Then, the minimum number of solutions of $f(x)g''(x) + f'(x)g'(x) = 0$ in $(-2, 2)$ is equal to____
[29-Jun-2022-Shift-2]

Answer: 4

Solution:

Solution:

Let $h(x) = f(x)g'(x) \to 5$ roots

$\because f(x)$ is even $\Rightarrow$

$f\left(\frac{1}{4}\right) = f\left(\frac{1}{2}\right) = f\left(-\frac{1}{2}\right) = f\left(\frac{1}{4}\right) = 0$

$g(x)$ is even $\Rightarrow g\left(\frac{3}{4}\right) = g\left(-\frac{3}{4}\right) = 0$

$g'(x) = 0$ has minimum one root

$h'(x)$ has at last 4 roots

Question53

The surface area of a balloon of spherical shape being inflated, increases at a constant rate. If initially, the radius of balloon is 3 units and after 5 seconds, it becomes 7 units, then its radius after 9 seconds is:
[24-Jun-2022-Shift-1]

Options:

A. 9

B. 10

C. 11

D. 12

Answer: A

Solution:

We know,

Surface area of balloon $(s) = 4\pi r^2$

$\therefore \frac{ds}{dt} = \frac{d}{dt}(4\pi r^2)$

$\Rightarrow \frac{ds}{dt} = 4\pi(2r) \times \frac{dr}{dt}$

$\Rightarrow \frac{ds}{dt} = 8\pi r \times \frac{dr}{dt}$

Given that, surface area of balloon is increasing in constant rate.

$\therefore \frac{ds}{dt} =$ constant $= k$ (Assume)

$\Rightarrow k = 8\pi r \cdot \frac{dr}{dt}$

$\Rightarrow \int k\,dt = \int 8\pi r\,dr$

$\Rightarrow kt = 8\pi \times \frac{r^2}{2} + c$

$\Rightarrow kt = 4\pi r^2 + c \ldots\ldots (1)$

Given at $t = 0$, radius $r = 3$

So, $0 = 4\pi(3^2) + c$

$\Rightarrow c = -36\pi$

$\therefore$ Equation (1) becomes

$kt = 4\pi r^2 - 36\pi$

Also given, at $t = 5$, radius $r = 7$

$\therefore k(5) = 4\pi(7)^2 - 36$

$\Rightarrow k = 32\pi$

$\therefore$ Equation (1) is

$32\pi t = 4\pi r^2 - 36\pi$

Now at $t = 9$

$\Rightarrow 32\pi(9) = 4\pi r^2 - 36\pi$

$\Rightarrow 8 \times 9 = r^2 - 9$

$\Rightarrow r^2 = 81 \Rightarrow r = 9$

Question54

For the function
$f(x) = 4\log_e(x - 1) - 2x^2 + 4x + 5, x > 1$, which one of the following is NOT correct?
[24-Jun-2022-Shift-1]

Options:

A. f is increasing in (1, 2) and decreasing in $(2, \infty)$

B. $f(x) = -1$ has exactly two solutions

C. $f'(e) - f''(2) < 0$

D. $f(x) = 0$ has a root in the interval $(e, e + 1)$

Answer: C

Solution:

Solution:

$f(x) = 4\log_e(x-1) - 2x^2 + 4x + 5, x > 1$

$f'(x) = \frac{4}{x-1} - 4(x-1)$

For $1 < x < 2 \Rightarrow f'(x) > 0$

For $x > 2 \Rightarrow f'(x) < 0$ (option A is correct)

$f(x) = -1$ has two solution (option B is correct)

$f(e) > 0$

$f(e+1) < 0$

$f(e) \cdot f(e+1) < 0$(option D is correct)

$f'(e) - f''(2) = \frac{4}{e-1} - 4(e-1) + 8 > 0$

Question55

If the tangent at the point (x_1, y_1) on the curve $y - x^3 + 3x^2 + 5$ passes through the origin, then $(x_{1,}, y_1)$ does NOT lie on the curve:
[24-Jun-2022-Shift-1]

Options:

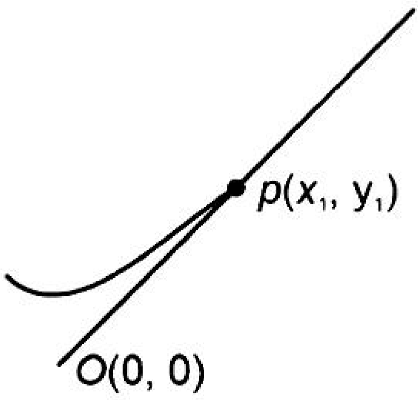
A. $x^2 + \frac{y^2}{81} = 2$

B. $\frac{y^2}{9} - x^2 = 8$

C. $y = 4x^2 + 5$

Answer: D

Solution:



Given curve,

$y = x^3 + 3x^2 + 5$

Slope of tangent,

$\frac{dy}{dx} = 3x^2 + 6x$

Slope of line joined by (x_1, y_1) and $(0, 0)$ is $= \frac{y_0 - 0}{x_1 - 0}$

$\therefore \frac{y_1}{x_1} = 3x_1^2 + 6x_1$

$\Rightarrow y_1 = 3x_1^3 + 6x_1^2 \ldots\ldots$ (1)

Point (x_1, y_1) is on the line,

$\therefore y_1 = x_1^3 + 3x_1^2 + 5 \ldots$ (2)

$\therefore 3x_1^3 + 6x_1^2 = x_1^3 + 3x_1^2 + 6$

$\Rightarrow 2x_1^3 + 3x_1^2 - 5 = 0$

$x = 1$ satisfy the equation

Now, $y_1 = x_1^3 + 3x_1^2 + 5$

$= 1 + 3 + 5$

$= 9$

$\therefore (x_1, y_1) = (1, 9)$

Option D does not satisfy point (1, 9).

Question56

The sum of absolute maximum and absolute minimum values of the function f (x) – 2x^2 + 3x – 2 | +sin x cos x in the interval [0, 1] is : [24-Jun-2022-Shift-1]

Options:

$\sin(1)\cos^2\left(\frac{1}{2}\right)$

B. $3 + \frac{1}{2}(1 + 2\cos(1))\sin(1)$

C. $5 + \frac{1}{2}(\sin(1) + \sin(2))$

D. $2 + \sin\left(\frac{1}{2}\right)\cos\left(\frac{1}{2}\right)$

Answer: B

Solution:

Solution:

$f(x) = |(2x-1)(x+2)| + \frac{\sin 2x}{2}$

$0 \le x < \frac{1}{2}$ $f(x) = (1-2x)(x+2) + \frac{\sin 2x}{2}$

$f'(x) = -4x - 3 + \cos 2x < 0$

For $x \ge \frac{1}{2}$: $f'(x) = 4x + 3 + \cos 2x > 0$

So, minima occurs at $x = \frac{1}{2}$

$f(x)|_{min} = \left|2\left(\frac{1}{2}\right)^2 + \frac{3}{2} - 2\right| + \sin\left(\frac{1}{2}\right)\cdot\cos\left(\frac{1}{2}\right)$

$= \frac{1}{2}\sin 1$

So, maxima is possible at $x = 0$ or $x = 1$

Now checking for $x = 0$ and $x = 1$, we can see it attains its maximum value at $x = 1$

$c.f(x)|_{max} = |2 + 3 - 2| + \frac{\sin 2}{2}$

$= 3 + \frac{1}{2}\sin 2$

Sum of absolute maximum and minimum value $= 3 + \frac{1}{2}(\sin 1 + \sin 2)$

Question57

If the absolute maximum value of the function $f(x) = (x^2 - 2x + 7)e^{(4x^3 - 12x^2 - 180x + 31)}$ in the interval $[-3, 0]$ is $f(\alpha)$, then :

[25-Jul-2022-Shift-1]

Options:

A. $\alpha = 0$

B. $\alpha = -3$

C. $\alpha \in (-1, 0)$

D. $\alpha \in (-3, -1]$

Answer: B

Given, $f(x) = \underbrace{(x^2 - 2x + 7)}_{f_1(x)} e^{\underbrace{(4x^3 - 12x^2 - 180x + 31)}_{f_2(x)}}$

$f_1(x) = x^2 - 2x + 7$

$f_1'(x) = 2x - 2$, so $f(x)$ is decreasing in $[-3, 0]$ and positive also

$f_2(x) = e^{4x^3 - 12x^2 - 180x + 31}$

$f_2'(x) = e^{4x^3 - 12x^2 - 180x + 31} \cdot 12x^2 - 24x - 180$

$= 12(x - 5)(x + 3)e^{4x^3 - 12x^2 - 180x + 31}$

So, $f_2(x)$ is also decreasing and positive in $\{-3, 0\}$

∴ absolute maximum value of $f(x)$ occurs at $x = -3$

Question58

The curve $y(x) = ax^3 + bx^2 + cx + 5$ touches the x-axis at the point P$(-2, 0)$ and cuts the y axis at the point Q, where y' is equal to 3 . Then the local maximum value of $y(x)$ is:
[25-Jul-2022-Shift-1]

Options:

A. $\frac{27}{4}$

B. $\frac{29}{4}$

C. $\frac{37}{4}$

D. $\frac{9}{2}$

Answer: A

Solution:

Solution:

$f(x) = y = ax^3 + bx^2 + cx + 5$ ... (i)

$\frac{dy}{dx} = 3ax^2 + 2bx + c$ ... (ii)

Touches x-axis at $P(-2, 0)$

$. \Rightarrow y|_{x=-2} = 0 \Rightarrow -8a + 4b - 2c + 5 = 0$ ...... (iii)

Touches x-axis at $P(-2, 0)$ also implies

$. \frac{dy}{dx}|_{x=-2} = 0 \Rightarrow 12a - 4b + c = 0$ ..... (iv)

$y = f(x)$ cuts y-axis at $(0, 5)$

Given, $. \frac{dy}{dx}|_{x=0} = c = 3$

From (iii), (iv) and (v)

$a = -\frac{1}{2}, b = -\frac{3}{4}, c = 3$

$\Rightarrow f(x) = \frac{-x^2}{2} - \frac{3}{4}x^2 + 3x + 5$

$f'(x) = \frac{-3}{2}x^2 - \frac{3}{2}x + 3$

$= \frac{-3}{2}(x + 2)(x - 1)$

$f'(x) = 0$ at $x = -2$ and $x = 1$

i.e., $\frac{27}{4}$

Question59

The sum of the maximum and minimum values of the function $f(x) = |5x - 7| + [x^2 + 2x]$ in the interval $\left[\frac{5}{4}, 2\right]$, where [t] is the greatest integer <t, is__
[25-Jul-2022-Shift-2]

Answer: 15

Solution:

$f(x) = |5x - 7| + [x^2 + 2x]$
$= |5x - 7| + [(x + 1)^2] - 1$
Critical points of
$f(x) = \frac{7}{5}, \sqrt{5} - 1, \sqrt{6} - 1, \sqrt{7} - 1, \sqrt{8} - 1, 2$
∴ Maximum or minimum value of $f(x)$ occur at critical points or boundary points
$\therefore f\left(\frac{5}{4}\right) = \frac{3}{4} + 4 = \frac{19}{4}$
$f\left(\frac{7}{5}\right) = 0 + 4 = 4$
as both $|5x - 7|$ and $x^2 + 2x$ are increasing in nature after $x = \frac{7}{5}$
$\therefore f(2) = 3 + 8 = 11$
$\therefore f\left(\frac{7}{5}\right)_{min} = 4$ and $f(2)_{max} = 11$
Sum is $4 + 11 = 15$

Question60

Let the area enclosed by the x-axis, and the tangent and normal drawn to the curve $4x^3 - 3xy^2 + 6x^2 - 5xy - 8y^2 + 9x + 14 = 0$ at the point $(-2, 3)$ be A. Then 8A is equal to
[25-Jul-2022-Shift-2]

Answer: 170

Solution:

Question61

Let $f(x) = \begin{cases} x^3 - x^2 + 10x - 7 & x \le 1 \\ -2x + \log_2(b^2 - 4) & x > 1. \end{cases}$.

Then the set of all values of b, for which f (x) has maximum value at x = 1, is :
[26-Jul-2022-Shift-1]

Options:

A. $(-6, -2)$

B. $(2, 6)$

C. $[-6, -2) \cup (2, 6]$

D. $[-\sqrt{6}, -2) \cup (2, \sqrt{6}]$

Answer: C

Solution:

Solution:

$f(x) = \begin{cases} x^3 - x^2 + 10x - 7 & x \le 1 \\ -2x + \log_2(b^2 - 4) & x > 1. \end{cases}$

If $f(x)$ has maximum value at $x = 1$ then $f(1+) \le f(1)$
$-2 + \log_2(b^2 - 4) \le 1 - 1 + 10 - 7$
$\log_2(b^2 - 4) \le 5$
$0 < b^2 - 4 \le 32$
(i) $b^2 - 4 > 0 \Rightarrow b \in (-\infty, -2) \cup (2, \infty)$
(ii) $b^2 - 36 \le 0 \Rightarrow b \in [-6, 6]$
Intersection of above two sets
$b \in [-6, -2) \cup (2, 6]$

Question62

If the maximum value of a, for which the function $f_a(x) = \tan^{-1}2x - 3ax + 7$ is non-decreasing in $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$, is $\bar{a}$, then $f_{\bar{a}}\left(\frac{\pi}{8}\right)$ is equal to
[26-Jul-2022-Shift-2]

Options:

A. $8 - \frac{9\pi}{4(9 + \pi^2)}$

B. $8 - \frac{4\pi}{9(4 + \pi^2)}$

C. $8\left(\frac{1+\pi^2}{9+\pi^2}\right)$

D. $8-\frac{\pi}{4}$

Answer: A

Solution:

Solution:

$f_a(x) = \tan^{-1}2x - 3ax + 7$

$\because f_a(x)$ is non-decreasing in $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$

$\therefore f_a'(x) \geq 0 \Rightarrow \frac{2}{1+4x^2} - 3a \geq 0$

$\Rightarrow 3a \leq \frac{2}{1+4x^2}$

So, $a_{max} = \frac{2}{3}\left(\frac{1}{1+4\times\frac{\pi^2}{36}}\right) = \frac{6}{9+\pi^2} = \bar{a}$

$\therefore f_{\bar{a}}\left(\frac{\pi}{8}\right) = 1 - 3\frac{\pi}{8}\cdot\frac{6}{9+\pi^2} + 7$

$= 8 - \frac{9\pi}{4(9+\pi^2)}$

Question63

Let $I = \int_{\pi/4}^{\pi/3}\left(\frac{8\sin x - \sin 2x}{x}\right)dx$. Then [27-Jul-2022-Shift-1]

Options:

A. $\frac{\pi}{2} < I < \frac{3\pi}{4}$

B. $\frac{\pi}{5} < I < \frac{5\pi}{12}$

C. $\frac{5\pi}{12} < I < \frac{\sqrt{2}}{3}\pi$

D. $\frac{3\pi}{4} < I < \pi$

Answer: C

Solution:

Solution:

I comes out around 1.536 which is not satisfied by any given options.

$\int_{\pi/4}^{\pi/3}\frac{8x-2x}{x}dx > I > \int_{\pi/4}^{\pi/3}\frac{8\sin x - 2x}{x}dx$

$\frac{\pi}{2} > I > \int_{\pi/4}^{\pi/3}\left(\frac{8\sin x}{x} - 2\right)dx$

$\frac{\sin x}{}$ is decreasing in $\left(\frac{\pi}{}, \frac{\pi}{}\right)$ so it attains maximum at x = $\frac{\pi}{}$

$$I > \int_{\pi/4}^{\pi/3}\left(\frac{8\sin x/3}{x/3} - 2\right)dx$$

$$I > \sqrt{3} - \frac{\pi}{6}$$

Question64

Let a function $f : \mathbb{R} \to \mathbb{R}$ be defined as :

$$f(x) = \begin{cases} \int_0^x (5 - |t-3|)dt & , x > 4 \\ x^2 + bx, & x \le 4 \end{cases}$$

where $b \in \mathbb{R}$. If f is continuous at $x = 4$, then which of the following statements is NOT true?
[27-Jul-2022-Shift-1]

Options:

A. f is not differentiable at $x = 4$

B. $f'(3) + f'(5) = \frac{35}{4}$

C. f is increasing in $\left(-\infty, \frac{1}{8}\right) \cup (8, \infty)$

D. f has a local minima at $x = \frac{1}{8}$

Answer: C

Solution:

Solution:

$\because f(x)$ is continuous at $x = 4$

$\Rightarrow f(4^-) = f(4^+)$

$\Rightarrow 16 + 4b = \int_0^4 (5 - |t-3|)dt$

$= \int_0^3 (2+t)dt + \int_3^4 (8-t)dt$

$.. = 2t + \frac{t^2}{2})_0^3 + 8t - \frac{t^2}{3}]_3^4$

$= 6 + \frac{9}{2} - 0 + (32 - 8) - \left(24 - \frac{9}{2}\right)$

$16 + 4b = 15$

$\Rightarrow b = \frac{-1}{4}$

$$\Rightarrow f(x) = \begin{cases} \int_0^x 5 - |t-3|dt & x > 4 \\ x^2 - \frac{x}{4} & x \le 4. \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 5 - |x-3| & x > 4 \\ 2x - \frac{1}{4} & x \le 4. \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 8 - x & x > 4 \\ \ldots \end{cases}$$

$f'(x) < 0 = x \in \left(-\infty, \frac{1}{8}\right) \cup (8, \infty)$

$f'(3) + f'(5) = 6 - \frac{1}{4} = \frac{35}{4}$

$f'(x) = 0 \Rightarrow x = \frac{1}{8}$ have local minima

Question65

Let M and N be the number of points on the curve $y^5 - 9xy + 2x = 0$, where the tangents to the curve are parallel to x-axis and y-axis, respectively. Then the value of M + N equals ________.
[27-Jul-2022-Shift-1]

Answer: 2

Solution:

Here equation of curve is
$y^5 - 9xy + 2x = 0$ ..... (i)
On differentiating: $5y^4\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} + 2 = 0$
$\therefore \frac{dy}{dx} = \frac{9y - 2}{5y^4 - 9x}$
When tangents are parallel to x-axis then $9y - 2 = 0$
$\therefore M = 1$.
For tangent perpendicular to x-axis
$5y^4 - 9x = 0$...... (ii)
From equation (i) and (ii) we get only one point.
$\therefore N = 1$.
$\therefore M + N = 2$

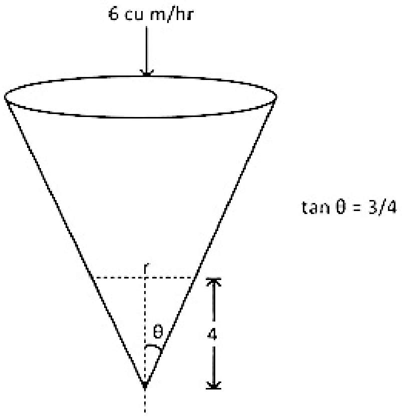
Question66

A water tank has the shape of a right circular cone with axis vertical and vertex downwards. Its semivertical angle is $\tan^{-1}\frac{3}{4}$. Water is poured in it at a constant rate of 6 cubic meter per hour. The rate (in square meter per hour), at which the wet curved surface area of the tank is increasing, when the depth of water in the tank is 4 meters, is ________.
[27-Jul-2022-Shift-2]

Answer: 5

Solution:

Solution:



$v = \frac{1}{3}\pi r^2 h.....$ (i)

And $\tan\theta = \frac{3}{4} = \frac{r}{h}.......$ (ii)

i.e. if $h = 4$, $r = 3$

$v = \frac{1}{3}\pi r^2\left(\frac{4r}{3}\right)$

$\frac{dv}{dt} = \frac{4\pi}{9}3r^2\frac{dr}{dt} \Rightarrow 6 = \frac{4\pi}{3}(9)\frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{2\pi}$

Curved area $= \pi r\sqrt{r^2 + h^2}$

$= \pi r\sqrt{r^2 + \frac{16r^2}{9}}$

$= \frac{5}{3}\pi r^2$

$\frac{dA}{dt} = \frac{10}{3}\pi r\frac{dr}{dt}$

$= \frac{10}{3}\pi \cdot 3 \cdot \frac{1}{2\pi}$

$= 5$

Question67

If the minimum value of $f(x) = \frac{5x^2}{2} + \frac{\alpha}{x^5}$, $x > 0$, is 14 , then the value of α is equal to:
[28-Jul-2022-Shift-1]

Options:

A. 32

B. 64

C. 128

D. 256

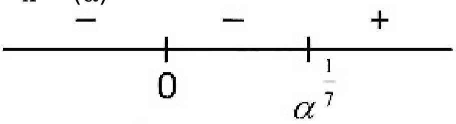
Answer: C

Solution:

$f(x)=\frac{5x^{2}}{2}+\frac{\alpha}{x^{5}}\{x>0\}$

$f'(x)=5x-\frac{5\alpha}{x^{6}}=0$

$\Rightarrow x=(\alpha)^{\frac{1}{7}}$



$f(x)_{min}=\frac{5(\alpha)^{\frac{2}{7}}}{2}+\frac{\alpha}{\alpha^{\frac{5}{7}}}=14$

$\frac{5}{2}\alpha^{\frac{2}{7}}+\alpha^{\frac{2}{7}}=14$

$\frac{7}{2}\alpha^{\frac{2}{7}}=14$

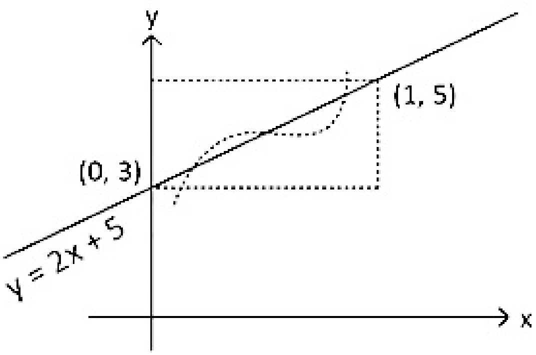
$\alpha=128$

Question68

Let $f:[0,1]\rightarrow R$ be a twice differentiable function in $(0,1)$ such that $f(0)-3$ and $f(1)-5$. If the line $y-2x+3$ intersects the graph of f at only two distinct points in $(0,1)$, then the least number of points $x\in(0,1)$, at which $f''(x)-0$, is _______.
[28-Jul-2022-Shift-1]

Answer: 2

Solution:



If a graph cuts $y=2x+5$ in $(0,1)$ twice then its concavity changes twice. $\therefore f'(x)=0$ at at least two points.

Question69

The function $f(x)=xe^{x(1-x)}$, $x\in\mathbb{R}$, is :
[28-Jul-2022-Shift-2]

Options:

A. increasing in $\left(-\frac{1}{2}, 1\right)$

B. decreasing in $\left(\frac{1}{2}, 2\right)$

C. increasing in $\left(-1, -\frac{1}{2}\right)$

D. decreasing in $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Answer: A

Solution:

Solution:
$f(x) = xe^{x(1-x)}, x \in R$
$f'(x) = xe^{x(1-x)} \cdot (1-2x) + e^{x(1-x)}$
$= e^{x(1-x)}[x - 2x^2 + 1]$
$= -e^{x(1-x)}[2x^2 - x - 1]$
$= -e^{x(1-x)}(2x+1)(x-1)$
$\therefore f(x)$ is increasing in $\left(-\frac{1}{2}, 1\right)$ and decreasing in $\left(-\infty, -\frac{1}{2}\right) \cup (1, \infty)$

Question70

The sum of the absolute maximum and absolute minimum values of the function $f(x) = \tan^{-1}(\sin x - \cos x)$ in the interval $[0, \pi]$ is : [28-Jul-2022-Shift-2]

Options:

A. 0

B. $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi}{4}$

C. $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) - \frac{\pi}{4}$

D. $\frac{-\pi}{12}$

Answer: C

Solution:

Solution:
$f(x) = \tan^{-1}(\sin x - \cos x), \quad [0, \pi]$
Let $g(x) = \sin x - \cos x$
$= \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$ and $x - \frac{\pi}{4} \in \left[\frac{-\pi}{4}, \frac{3\pi}{4}\right]$
$\therefore g(x) \in [-1, \sqrt{2}]$
and $\tan^{-1}x$ is an increasing function
$\therefore f(x) \in [\tan^{-1}(-1), \tan^{-1}\sqrt{2}]$

$\in \left[-\frac{\pi}{4}, \tan^{-1}\sqrt{2}\right]$

$\therefore$ Sum of f_{max} and $f_{min} = \tan^{-1}\sqrt{2} - \frac{\pi}{4}$

$= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) - \frac{\pi}{4}$

Question71

Let $f(x) = 3^{(x^2-2)^3+4}$, $x \in R$. Then which of the following statements are true?
P : $x = 0$ is a point of local minima of f
Q : $x = \sqrt{2}$ is a point of inflection of f
R : f' is increasing for $x > \sqrt{2}$
[29-Jul-2022-Shift-1]

Options:

A. Only P and Q

B. Only P and R

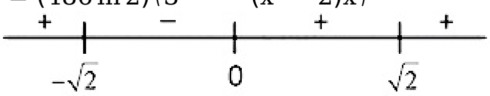
C. Only Q and R

D. All P, Q and R

Answer: D

Solution:

Solution:
$f(x) = 3^{(x^2-2)^3+4}$, $x \in R$
$f(x) = 81.3^{(x^2-2)^3}$
$f'(x) = 81.3^{(x^2-2)^3} \ln 2.3(x^2-2)2x$
$= (486\ln 2)\left(3^{(x^2-2)^3}(x^2-2)x\right)$



$\Rightarrow x = 0$ is the local minima.
$f''(x) = (486\ln 2)\left(3^{(x^2-2)^3} \cdot (x^2-2)(5x^2-2+6x^2\ln 3(x^2-2))\right)$
$f''(x) = 0 \ \ x = \sqrt{2}$
$f''\left(\sqrt{2^+}\right) > 0$
$f''\left(\sqrt{2^-}\right) < 0$
$\Rightarrow x = \sqrt{2}$ is point of inflection
$f''(x) > 0 \ \forall x > \sqrt{2}$
$\Rightarrow f'(x)$ is increasing for $x > \sqrt{2}$

Question72

If the tangent to the curve $y = x^3 - x^2 + x$ at the point (a, b) is also tangent to the curve $y = 5x^2 + 2x - 25$ at the point (2, −1), then |2a + 9b| is equal to

[29-Jul-2022-Shift-2]

Answer: 195

Solution:

Solution:
Slope of tangent to curve $y = 5x^2 + 2x - 25$
$= m = \left(\frac{dy}{dx}\right)_{at(2,-1)} = 22$
$\therefore$ Equation of tangent : $y + 1 = 22(x - 2)$
$\therefore\ y = 22x - 45$.
Slope of tangent to $y = x^3 - x^2 + x$ at point (a, b)
$= 3a^2 - 2a + 1$
$3a^2 - 2a + 1 = 22$
$3a^2 - 2a - 21 = 0$
$\therefore\ a = 3 \text{ or } -\frac{7}{3}$
Also $b = a^3 - a^2 + a$
Then $(a, b) = (3, 21)$ or $\left(-\frac{7}{3}, -\frac{151}{9}\right)$.
$\left(-\frac{7}{3}, -\frac{151}{9}\right)$ does not satisfy the equation of tangent
$\therefore\ a = 3, b = 21$
$\therefore |2a + 9b| = 195$

Question73

Let a be an integer, such that all the real roots of the polynomial $2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$ lie in the interval (a, a + 1). Then, |a| is equal to ______.
[26 Feb 2021 Shift 2]

Answer: 2

Solution:

Let $f(x) = 2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$
Using hit and trial method,
$f(-2) = -34 < 0$ and $f(-1) = 3 > 0$
Hence, $f(x)$ has a root in $(-2, -1)$.
Again,
$f'(x) = 10x^4 + 20x^3 + 30x^2 + 20x + 10$
$= 10x^2\left(x^2 + 2x + 3 + \frac{2}{x} + \frac{1}{x^2}\right)$

$$= 10x^2\left[\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)+3\right]$$
$$= 10x^2\left[\left(x+\frac{1}{x}\right)^2+1+2\left(x+\frac{1}{x}\right)\right]$$
$$= 10x^2\left[\left(x+\frac{1}{x}+1\right)^2\right] > 0\ \forall x$$

⇒$f(x)$ is strictly increasing function, since degree of $f(x)$ is odd.
∴ It has exactly one real root.
Therefore, $f(x)$ has atleast one root in
⇒ $|a| = |-2| = 2$

Question74

If Rolle's theorem holds for the function $f(x) = x^3 - ax^2 + bx + 4,\ x \in [1, 2]$ with $f'\left(\frac{4}{3}\right) = 0$, then ordered pair (a, b) is equal to
[25 Feb 2021 Shift 1]

Options:

A. (5, 8)

B. (−5, 8)

C. (5, −8)

D. (−5, −8)

Answer: A

Solution:

Solution:
Given, $f(x) = x^3 - ax^2 + bx + 4,\ x \in [1, 2]$
Here, $f(1) = f(2)$
⇒ $1 - a + b - 4 = 8 - 4a + 2b - 4$
⇒$3a - b = 7$ . . . (i)
Also, $f'(x) = 3x^2 - 2ax + b$
According to the question, $f'\left(\frac{4}{3}\right) = 0$
⇒ $3 \times \left(\frac{4}{3}\right)^2 - 2a\left(\frac{4}{3}\right) + b = 0$
⇒$-8a + 3b = -16$ . . . (ii)
From Eqs. (i) and (ii),
$a = 5, b = 8$
$(a, b) = (5, 8)$
∴ $(a, b) = (5, 8)$

Question75

Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} -55x & \text{if} \quad x < -5 \\ 2x^3 - 3x^2 - 120x & \text{if} \quad -5 \le x < 4 \\ 2x^3 - 3x^2 - 36x - 336 & \text{if} \quad x \ge 4. \end{cases}$$

Let A = { x ∈ R : f is increasing }. Then, A is equal to [24 Feb 2021 Shift 2]

Options:

A. $(-\infty, -5) \cup (4, \infty)$

B. $(-5, \infty)$

C. $(-\infty, -5) \cup (-4, \infty)$

D. $(-5, -4) \cup (4, \infty)$

Answer: D

Solution:

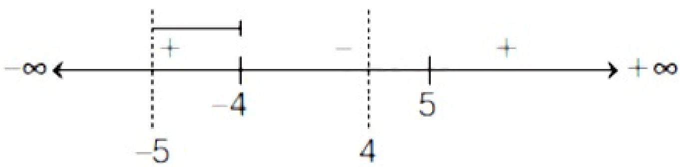
Solution:

Given, $f(x) = \begin{cases} -55x & x < -5 \\ 2x^3 - 3x^2 - 120x & -5 \le x < 4 \\ 2x^3 - 3x^2 - 36x + 10 & x \ge 4. \end{cases}$

$\therefore\ f'(x) = \begin{cases} -55 & x < -5 \\ 6(x^2 - x - 20) & -5 \le x < 4 \\ 6(x^2 - x - 6) & x \ge 4. \end{cases}$

$f'(x) = \begin{cases} -55 & x < -5 \\ 6(x-5)(x+4) & -5 \le x < 4 \\ 6(x-3)(x+2) & x \ge 4. \end{cases}$

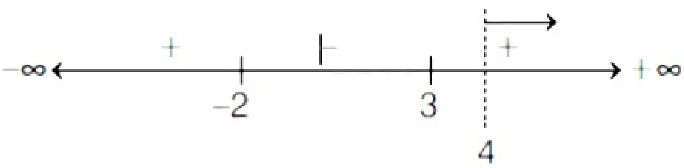
For f to be increasing, $f'(x) > 0$
Now, $f'(x) = -55$ is always less than zero.
$f'(x) = 6(x-5)(x+4) < 0, -5 \le x < 4$
Critical points $= 5, -4$



$[x \in (-5, -4)$

and $f'(x) = 6(x-3)(x+2) < 0, x \ge 4$

Critical point, $= 3, -2$



$x \in (4, \infty)$ ... (ii)
From Eqs. (i) and (ii), $f(x)$ is increasing in $x \in (-5, -4) \cup (4, \infty)$

The triangle of maximum area that can be inscribed in a given circle of radius r is
[26 Feb 2021 Shift 2]

Options:

A. an isosceles triangle with base equal to 2r

B. an equilateral triangle of height $\frac{2r}{3}$

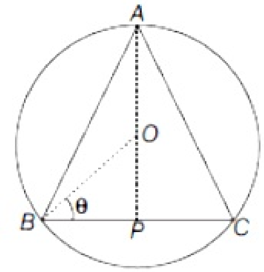
C. an equilateral triangle having each of its side of length $\sqrt{3}r$

D. a right angle triangle having two of its sides of length 2r and r

Answer: C

Solution:

Solution:
Let a $\triangle ABC$ inscribed in a circle with centre O and radius r.



Let $\angle OBC = \theta$
Now, area of $\triangle ABC = \frac{1}{2} \times$ Base × Height
$A = \frac{1}{2} \times (BC) \times (AP) \ldots$ (i)
Now, $BC = 2BP$
Consider ΔOBP, where $OB = r$
Then, $BP = r\cos\theta$
Hence, $BC = 2r\cos\theta$
Again, $AP = AO + OP$
where, $AO = r$
Consider ΔOBP, where $OB = r$
Then, $OP = r\sin\theta$
$\Rightarrow AP = r + r\sin\theta$
From Eq. (i), we get
Area $= \frac{1}{2} \times (2r\cos\theta) \times (r + r\sin\theta)$
$A = r^2\cos\theta(1 + \sin\theta)$
Now, $\frac{dA}{d\theta} = r^2(-\sin\theta)(1 + \sin\theta) + r^2\cos^2\theta$
$= r^2(\cos^2\theta - \sin\theta - \sin^2\theta)$
$= r^2(1 - 2\sin^2\theta - \sin\theta)$
$= r^2(1 + \sin\theta)(1 - 2\sin\theta)$
Equate $\frac{dA}{d\theta} = 0$
$\Rightarrow r^2(1 + \sin\theta)(1 - 2\sin\theta) = 0$
$r = r^2(1 + \sin\theta)(1 - 2\sin\theta) = 0$
$\Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$
Now, $\frac{d^2A}{d\theta^2} < 0$, when $\theta = \frac{\pi}{6}$
$\Rightarrow$A is maximum, when $\theta = \frac{\pi}{6}$
$\therefore$ Maximum area $= r^2\cos\left(\frac{\pi}{6}\right)\left(1 + \sin\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{4}r^2$

Consider $\triangle ABP$,
$(AB)^2 = (AP)^2 + (BP)^2$
$= \left(\frac{3}{2}r\right)^2 + \left(\frac{\sqrt{3}}{2}r\right)^2$ $[\because BP = r\cos\theta]$
$= \frac{9}{4}r^2 + \frac{3}{4}r^2 = 3r^2$
$\Rightarrow AB = \sqrt{3}r$
Hence, the $\triangle ABC$ is an equilateral triangle with side $\sqrt{3}$.

Question77

The maximum slope of the curve $y = \frac{1}{2}x^4 - 5x^3 + 18x^2 - 19x$ occurs at the point
[26 Feb 2021 Shift 1]

Options:

A. (2, 2)

B. (0, 0)

C. (2, 9)

D. $\left(3, \frac{21}{2}\right)$

Answer: A

Solution:

Solution:
Given, curve is $y = \frac{1}{2}x^4 - 5x^3 + 18x^2 - 19x$ . . . (i)
First, find the slope of given curve i.e. dy/dx,
Differentiate Eq. (i),
$\frac{dy}{dx} = \frac{1}{2}(4x^3) - 5(3x^2) + 18(2x) - 19$
$= 2x^3 - 15x^2 + 36x - 19$
Now, let $f(x) = 2x^3 - 15x^2 + 36x - 19$ is slope of the curve and find its maximum value as follows,
$f'(x) = 2(3x^2) - 15(2x) + 36 = 6x^2 - 30x + 36$
Equate $f'(x) = 0$ and solve for 'x'',
$6x^2 - 30x + 36 = 0$
$\Rightarrow x^2 - 5x + 6 = 0$
$\Rightarrow x^2 - 3x - 2x + 6 = 0$
$\Rightarrow (x-3)(x-2) = 0$
$\Rightarrow x = 2$ and 3
Now, $f''(x) = \frac{d}{dx}(6x^2 - 30x + 36)$
$= 12x - 30$
Then, $f''(2) = 12(2) - 30 = 24 - 30$
$= -6 < 0$
and $f''(3) = 12(3) - 30 = 6 > 0$
$\because f''(2) < 0$, this implies $2'$ is point of maxima.
$\therefore$ At $x = 2$, slope will be maximum.
Since, at $x = 2$, slope will be maximum, then y-coordinate will be,
$y = \frac{1}{2}(2)^4 - 5(2)^3 + 18(2)^2 - 19(2)$
$= 8 - 40 + 72 - 38 = 72 - 70 = 2$
$\therefore$ Maximum slope occurs at point (2, 2).

Question78

The shortest distance between the line $x - y = 1$ and the curve $x^2 = 2y$ is [25 Feb 2021 Shift 2]

Options:

A. $\frac{1}{\sqrt{2}}$

B. $\frac{1}{2\sqrt{2}}$

C. 0

D. $\frac{1}{2}$

Answer: B

Solution:

Solution:
Let (x, y) be any arbitrary point on curve $x^2 = 2y$ and find the tangent line equation at this point, such that tangent line at (x, y) is parallel to line $x - y = 1$.
To find tangent equation, differentiate the following equation so that we can find slope,
$x^2 - 2y = 0$
$2x - 2\frac{dy}{dx} = 0$ gives $\frac{dy}{dx} = x$
Slope (say m_1) $= x$
Also, slope of line $x - y = 1$ or $y = x - 1$ is 1 (say m_2). Since, $x - y = 1$ and tangent line is parallel, therefore, their slope be equal.
Hence, $m_1 = m_2$ gives, $x = 1$
Put $x = 1$ in Eq. (i), we get $y = 1/2$
Thus, $(x, y) = \left(1, \frac{1}{2}\right)$
Perpendicular distance between line $x - y = 1$ and point $\left(1, \frac{1}{2}\right)$ is given as,

$$P = \left| \frac{(1)(1) + \left(\frac{1}{2}\right)(-1) - 1}{\sqrt{(1)^2 + (-1)^2}} \right|$$

$$= \left| \frac{-1}{2\sqrt{2}} \right|$$

$$= \frac{1}{2\sqrt{2}} \quad \therefore \text{ using}$$

(∴ using perpendicular distance formula)

Question79

If the curves $x = y^4$ and $xy = k$ cut at right angles, then $(4k)^6$ is equal to .... [25 Feb 2021 Shift 2]

Answer: 4

Solution:

If the curves cut at right angle, then product of slopes will be -1.
First curve $x = y^4$
Differentiate it, we get

$$1 = 4y^3 \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{4y^3}$$

$\Rightarrow$ Slope of first curve $(m_1) = \frac{1}{4{y_1}^3}$ [at point (x_1, y_1)]

Second curve $xy = k$
Differentiate it, $0 = x\frac{dy}{dx} + y$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

Slope of second curve $(m_2) = \frac{-y_1}{x_1}$ [at (x_1, y_1)]

$$\Rightarrow m_1 \cdot m_2 = -1$$

$$\Rightarrow \frac{1}{4{y_1}^3}\left(\frac{-y_1}{x_1}\right) = -1 \Rightarrow \frac{-1}{4{y_1}^2 x_1} = -1$$

$$\Rightarrow \frac{-1}{4(y_1)^6} = -1$$

$$\Rightarrow {y_1}^6 = \frac{1}{4}$$

Also, $x_1 y_1 = k$, using $x_1 = {y_1}^4$, we get $k = {y_1}^5$ or $k^6 = (y_1)^{30}$

$\because {y_1}^6 = \frac{1}{4}$, then ${y_1}^{30} = \left(\frac{1}{4}\right)^5$

$$\therefore (4k)^6 = 4^6 \cdot k^6 = 4^6 (y_1)^{30} = 4^6\left(\frac{1}{4}\right)^5 = 4$$

$$\therefore (4k)^6 = 4$$

Question80

Let $f(x)$ be a polynomial of degree 6 in x, in which the coefficient of x^6 is unity and it has extrema at $x = -1$ and $x = 1$. If $\lim_{x \to 0} \frac{f(x)}{x^3} = 1$, then $5f(2)$ is equal to ....
[25 Feb 2021 Shift 1]

Answer: 144

Solution:

$$f(x) = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

As, $\lim_{x \to 0} \frac{f(x)}{x^3} = 1$ non-zero finite
So, $d = e = f = 0$ and $f(x) = x^3(x^3 + ax^2 + bx + c)$
Hence, $\lim_{x \to 0} \frac{f(x)}{x^3} = c = 1$
Now, as $f(x) = x^6 + ax^5 + bx^4 + x^3$
and $f'(x) = 0$ at $x = 1$ and $x = -1$

i.e. $f'(x) = 6x^5 + 5ax^4 + 4bx^3 + 3x^2$
Now, $f'(1) = 0$
$\Rightarrow 6 + 5a + 4b + 3 = 0$
$\Rightarrow 5a + 4b = -9 \ldots$ (i)
and $f'(-1) = 0$
$\Rightarrow -6 + 5a - 4b + 3 = 0$
$\Rightarrow 5a - 4b = 3 \ldots$ (ii)
From Eqs. (i) and (ii),
$a = -3/5$ and $b = -3/2$
$\therefore f(x) = x^6 - \frac{3}{5}x^5 - \frac{3}{2}x^4 + x^3$
$\therefore 5f(2) = 5\left[2^6 - \frac{3}{5}(2)^5 - \frac{3}{2}(2)^4 + (2)^3\right]$
$= 5\left[64 - \frac{3 \times 32}{5} - \frac{3 \times 16}{2} + 8\right]$
$= 320 - 96 - 120 + 40$
$= 144$

Question81

If the tangent to the curve $y = x^3$ at the point $P(t, t^3)$ meets the curve again at Q, then the ordinate of the point which divides PQ internally in the ratio 1 : 2 is :
24 Feb 2021 Shift 1

Options:

A. $-2t^3$

B. 0

C. $-t^3$

D. $2t^3$

Answer: A

Solution:

Solution:
Slope of tangent at $P(t, t^3) = \frac{dy}{dx}]_{(t, t^3)} = (3x^2)^{x=t} = 3t^2$
So, equation tangent at $P(t, t^3)$:
$y - t^3 = 3t^2(x - t)$
For point of intersection with $y = x^3$
$x^3 - t^3 = 3t^2x - 3t^3$
$\Rightarrow (x - t)(x^2 + xt + t^2) = 3t^2(x - t)$
For $x \neq t$
$x^2 + xt + t^2 = 3t^2$
$\Rightarrow x^2 + xt - 2t^2 = 0 \Rightarrow (x - t)(x + 2t) = 0$
So, for $Q : x = -2t, Q(-2t, -8t^3)$
ordinate of required point :
$\frac{2t^3 - 8t^3}{2 + 1} = -2t^3$

The function $f(x) = \frac{4x^3 - 3x^2}{6} - 2\sin x + (2x - 1)\cos x$:
24 Feb 2021 Shift 1

Options:

A. increases in $\left[\frac{1}{2}, \infty\right)$

B. increases in $\left(-\infty, \frac{1}{2}\right]$

C. decreases in $\left[\frac{1}{2}, \infty\right)$

D. decreases in $\left(-\infty, \frac{1}{2}\right]$

Answer: A

Solution:

Solution:
Given that
$f(x) = \frac{4x^3 - 3x^2}{6} - 2\sin x + (2x - 1)\cos x$
$f'(x) = (2x^2 - x) - 2\cos x + 2\cos x$
$-\sin x(2x - 1)$
$= (2x - 1)(x - \sin x)$
For $x > 0$, $x - \sin x > 0$
$x < 0$, $x - \sin x < 0$
For $x \in (-\infty, 0] \cup \left[\frac{1}{2}, \infty\right)$, $f'(x) \geq 0$
For $x \in \left[0, \frac{1}{2}\right]$, $f'(x) \leq 0$
$\Rightarrow f(x)$ increases in $\left[\frac{1}{2}, \infty\right)$.

Question83

The minimum value of α for which the equation $\frac{4}{\sin x} + \frac{1}{1 - \sin x} = \alpha$ has at least one solution in $\left(0, \frac{\pi}{2}\right)$ is
24 Feb 2021 Shift 1

Answer: 9

Solution:

Let $f(x) = \frac{4}{\sin x} + \frac{1}{1 - \sin x}$
$\Rightarrow f'(x) = 0 \Rightarrow \sin x = 2/3$

$\therefore f(x)_{min} = \frac{4}{2/3} + \frac{1}{1-2/3} = 9$
$f(x)_{max.} \to \infty$
f (x) is continuous function
$\therefore \alpha_{min} = 9$

Question84

Let f be a real valued function, defined on $R - \{-1, 1\}$ and given by
$f(x) = 3\log_e \left| \frac{x-1}{x+1} \right| - \frac{2}{x-1}$
Then, in which of the following intervals, function f (x) is increasing?
[16 Mar 2021 Shift 2]

Options:

A. $(-\infty, -1) \cup \left(\left[\frac{1}{2}, \infty \right) - \{1\} \right)$

B. $(-\infty, \infty) - \{-1, 1\}$

C. $\left(-1, \frac{1}{2} \right]$

D. $\left(-\infty, \frac{1}{2} \right] - [-1]$

Answer: A

Solution:

Given, $f(x) = 3\log_e \left| \frac{x-1}{x+1} \right| - \frac{2}{x-1}$

$\Rightarrow f'(x) = \frac{3}{\left(\frac{x-1}{x+1} \right)} \cdot \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2} + \frac{2}{(x-1)^2}$

$\Rightarrow f'(x) = 3\left(\frac{x+1}{x-1} \right)\left[\frac{2}{(x+1)^2} \right] + \frac{2}{(x-1)^2}$

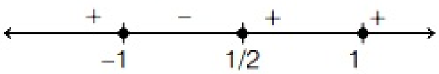
$\Rightarrow f'(x) = \left(\frac{2}{x-1} \right)\left(\frac{3}{x+1} + \frac{1}{x-1} \right)$

$\Rightarrow f'(x) = \left(\frac{2}{x-1} \right)\left[\frac{3x-3+x+1}{(x-1)(x+1)} \right]$

$\Rightarrow f'(x) = \frac{2\cdot 2\cdot (2x-1)}{(x-1)^2(x+1)}$

So, $f'(x) = \frac{4(2x-1)}{(x-1)^2(x+1)}$

Now,



For f (x) to be an increasing function, $f'(x) > 0$.
And $f'(x) > 0$ at $x \in (-\infty, -1) \cup \left[\frac{1}{2}, \infty \right)$
But domain of f (x) is $x \in (-\infty, -1) \cup (1, \infty)$
So, $f'(x) > 0$ at $x \in (-\infty, -1) \cup \left[\frac{1}{2}, 1 \right) \cup (1, \infty)$

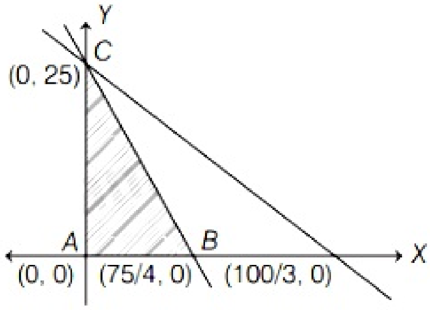
Question85

The maximum value of z in the following equation $z = 6xy + y^2$, where $3x + 4y \le 100$ and $4x + 3y \le 75$ for $x \ge 0$ and $y \ge 0$ is
[17 Mar 2021 Shift 1]

Answer: 904

Solution:

$z = 6xy + y^2$
$3x + 4y \le 100$
$4x + 3y \le 75$
$x, y \ge 0$
$z = y(6x + y)$



z will be maximum at the corner points.

$x \le \dfrac{75 - 3y}{4}$

$z = y(6x + y)$

$z \le y\left[6\left(\dfrac{75 - 3y}{4}\right) + y\right]$

$z \le \dfrac{1}{2}(225 - 7y^2)$

$(225y - 7y^2)$ is a quadratic in y whose maximum value is $\dfrac{-D}{4a}$.

Here, $D = \dfrac{(225)^2 - 4 \cdot 0(-7)}{4(-7)}$

$\therefore\ z \le \dfrac{(225)^2}{2 \cdot 4 \cdot 7} = \dfrac{50625}{56} \simeq 904$

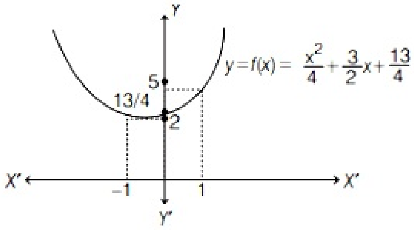
Question86

Let $f : [-1, 1] \to R$ be defined as $f(x) = ax^2 + bx + c$ for all $x \in [-1, 1]$, where $a, b, c \in R$, such that $f(-1) = 2$, $f'(-1) = 1$ and for $x \in (-1, 1)$ the maximum value of $f''(x)$ is $\frac{1}{2}$. If $f(x) \le \alpha$, $x \in [-1, 1]$, then the least value of α is equal to
[17 Mar 2021 Shift 2]

Answer: 5

Solution:

Given, $f : [-1, 1] \rightarrow R$
and $f(x) = ax^2 + bx + c$
$f(-1) = a - b + c = 2$ (given)...(i)
$f'(-1) = -2a + b = 1$ (given)...(ii)
$f''(x) = 2a$
$\therefore f_{max}''(x) = 2a$
Also, given maximum value of $f''(x) = \frac{1}{2}$
i.e. $2a = \frac{1}{2} \Rightarrow a = \frac{1}{4}$
From Eq. (ii), $b = \frac{3}{2}$
From Eq. (i), $c = \frac{13}{4}$
$\therefore f(x) = \frac{x^2}{4} + \frac{3}{2}x + \frac{13}{4}$
Here, $f(-1) = \frac{1}{4} - \frac{3}{2} + \frac{13}{4} = 2$
and $f(1) = \frac{1}{4} + \frac{3}{2} + \frac{13}{4} = 5$



For $x \in [-1, 1]$
$f(x) \in [2, 5]$
$\therefore$ Least value of α is 5 .

Question87

The range of $a \in R$ for which the function $f(x) = (4a - 3)(x + \log_e 5) + 2(a - 7)\cot\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)$, $x \neq 2n\pi$, $n \in N$, has critical points, is
[16 Mar 2021 Shift 1]

Options:

A. $(-3, 1)$

B. $\left[-\frac{4}{3}, 2\right]$

C. $[1, \infty)$

D. $(-\infty, -1]$

Answer: B

Solution:

Given, $f(x) = (4a-3)(x+\log_e 5) + 2(a-7)\cot\frac{x}{2} \cdot \sin^2\frac{x}{2}$

$\Rightarrow f(x) = (4a-3)(x+\log_e 5) + 2(a-7)\cos\frac{x}{2}\sin\frac{x}{2}$

$\Rightarrow f(x) = (4a-3)(x+\log_e 5) + (a-7)\sin x$

$\Rightarrow f'(x) = (4a-3)(1+0) + (a-7)\cos x$

$\Rightarrow f'(x) = (4a-3) + (a-7)\cos x$

When $f'(x) = 0$, $(4a-3) + (a-7)\cos x = 0$

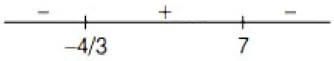
$\Rightarrow \cos x = \frac{4a-3}{7-a}$

As, $-1 \le \cos x \le 1$

So, $-1 \le \frac{4a-3}{7-a} \le 1$

$\Rightarrow \frac{4a-3}{7-a} + 1 \ge 0 \Rightarrow \frac{4a-3+7-a}{7-a} \ge 0$

$\Rightarrow \frac{3a+4}{7-a} \ge 0$



$a \in [-4/3, 7)$ ...(i)

Now, $\frac{4a-3}{7-a} - 1 \le 0$

$\Rightarrow \frac{4a-3-7+a}{7-a} \le 0$

$\Rightarrow \frac{5a-10}{7-a} \le 0$

$a \in (-\infty, 2] \cup (7, \infty)$...(ii)

From Eqs. (i) and (ii),

$a \in \left[-\frac{4}{3}, 2\right]$

Question88

Let $f(x) = 3\sin^4 x + 10\sin^3 x + 6\sin^2 x - 3, x \in \left[-\frac{\pi}{6}, \frac{\pi}{2}\right]$. Then, f is :
[25 Jul 2021 Shift 1]

Options:

A. increasing in $\left(-\frac{\pi}{6}, \frac{\pi}{2}\right)$

B. decreasing in $\left(0, \frac{\pi}{2}\right)$

C. increasing in $\left(-\frac{\pi}{6}, 0\right)$

D. decreasing in $\left(-\frac{\pi}{6}, 0\right)$

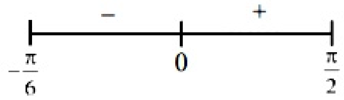
Answer: D

Solution:

Solution:

$f(x) = 3\sin^4 x + 10\sin^3 x + 6\sin^2 x - 3, x \in \left[-\frac{\pi}{6}, \frac{\pi}{2}\right]$.

$= 6 \sin x \cos x(2\sin^2 x + 5 \sin x + 2)$
$= 6 \sin x \cos x(2 \sin x + 1)(\sin + 2)$



decreasing in $\left(-\frac{\pi}{6}, 0\right)$

Question89

Let $f : R \to R$ be defined as $f(x) = \begin{cases} -\frac{4}{3}x^3 + 2x^2 + 3x & , x > 0 \\ 3xe^x & , x \le 0 \end{cases}$. Then f is increasing function in the interval [22 Jul 2021 Shift 2]

Options:

A. $\left(-\frac{1}{2}, 2\right)$

B. $(0, 2)$

C. $\left(-1, \frac{3}{2}\right)$

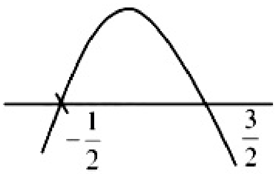
D. $(-3, -1)$

Answer: C

Solution:

Solution:

$f'(x) \begin{cases} -4x^3 + 4x + 3 & x > 0 \\ 3e^x(1 + x) & x \le 0 \end{cases}$

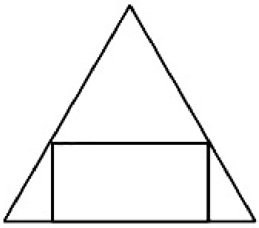


For $x > 0$, $f'(x) = -4x^2 + 4x + 3$
$f(x)$ is increasing in $\left(-\frac{1}{2}, \frac{3}{2}\right)$
For $x \le 0$, $f'(x) = 3e^x(1 + x)$
$f'(x) > 0 \ \forall x \in (-1, 0)$
$\Rightarrow f(x)$ is increasing in $(-1, 0)$
So, in complete domain, $f(x)$ is increasing in $\left(-1, \frac{3}{2}\right)$

Question90

If a rectangle is inscribed in an equilateral triangle of side length $2\sqrt{2}$

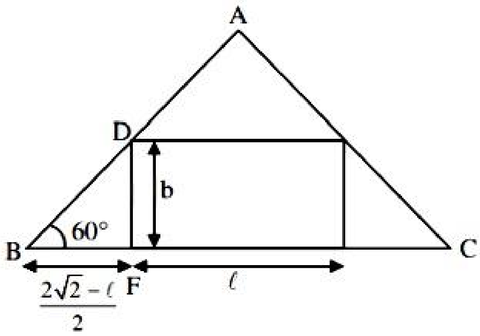
as shown in the figure, then the square of the largest area of such a rectangle is _______.



[25 Jul 2021 Shift 2]

Answer: 3

Solution:



In ΔDBF

$\tan 60° = \frac{2b}{2\sqrt{2}-l} \Rightarrow b = \frac{\sqrt{3}(2\sqrt{2}-l)}{2}$

$A =$ Area of rectangle $= l \times b$

$A = l \times \frac{\sqrt{3}}{2}(2\sqrt{2}-l)$

$\frac{dA}{dl} = \frac{\sqrt{3}}{2}(2\sqrt{2}-l) - \frac{l \cdot \sqrt{3}}{2} = 0$

$l = \sqrt{2}$

$A = l \times b = \sqrt{2} \times \frac{\sqrt{3}}{2}(\sqrt{2}) = \sqrt{3}$

$\Rightarrow A^2 = 3$

Question91

Let 'a' be a real number such that the function $f(x) = ax^2 + 6x - 15, x \in R$ is increasing in $\left(-\infty, \frac{3}{4}\right)$ and decreasing in $\left(\frac{3}{4}, \infty\right)$. Then the function $g(x) = ax^2 - 6x + 15, x \in R$ has a:
[20 Jul 2021 Shift 1]

Options:

A. local maximum at $x = -\frac{3}{4}$

B. local minimum at $x = -\frac{3}{4}$

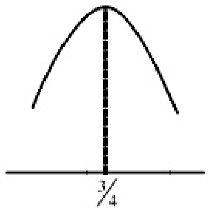
C. local maximum at $x = \frac{3}{4}$

D. local minimum at $x = \frac{3}{4}$

Answer: A

Solution:

Solution:



$\frac{-B}{2A} = \frac{3}{4}$

$\Rightarrow \frac{-(6)}{2a} = \frac{3}{4}$

$\Rightarrow a = \frac{-6 \times 4}{6} \Rightarrow a = -4$

$\therefore g(x) = 4x^2 - 6x + 15$

Local max. at $x = \frac{-B}{2A} = -\frac{(-6)}{2(-4)}$

$= \frac{-3}{4}$

Question92

The sum of all the local minimum values of the twice differentiable function f : R → R defined by $f(x) = x^3 - 3x^2 - \frac{3f''(2)}{2}x + f''(1)$ is :
[20 Jul 2021 Shift 2]

Options:

A. −22

B. 5

C. −27

D. 0

Answer: C

Solution:

Solution:

$f(x) = x^3 - 3x^2 - \frac{3}{2}f''(2)x + f''(1)$ ....(i)

$f'(x) = 3x^2 - 6x - \frac{3}{2}f''(2)$ ......(ii)

$f''(x) = 6x - 6$......... (iii)

Now is 3^{rd} equation

$f''(2) = 12 - 6 = 6$

$f'(x) = 3x^2 - 6x - \frac{3}{2}f''(2)$

$f'(x) = 3x^2 - 6x - \frac{3}{2} \times 6$

$f'(x) = 3x^2 - 6x - 9$

$f'(x) = 0$

$3x^2 - 6x - 9 = 0$

$\Rightarrow x = -1 \& 3$

Use (iii)

$f''(x) = 6x - 6$

$f''(-1) = -12 < 0$ maxima

$f''(3) = 12 > 0$ minima.

Use (i)

$f(x) = x^3 - 3x^2 - \frac{3}{2}f''(2)x + f''(1)$

$f(x) = x^3 - 3x^2 - \frac{3}{2} \times 68x + 0$

$f(x) = x^3 - 3x^2 - 9x$

$f(3) = 27 - 27 - 9 \times 3 = -27$

Question93

The number of real roots of the equation $e^{4x} + 2e^{3x} - e^x - 6 = 0$ is [2021, 31 Aug. Shift-1]

Options:

A. 2

B. 4

C. 1

D. 0

Answer: C

Solution:

Solution:

$f(x) = e^{4x} + 2e^{3x} - e^x - 6$

$f'(x) = 4e^{4x} + 6e^{3x} - e^x$

$= e^x(4e^{3x} + 6e^{2x} - 1)$

Let $g(x) = e^x$

$h(x) = 4e^{3x} + 6e^{2x} - 1$

$g(x) > 0, \forall x \in R$

$h'(x) = 12e^{3x} + 12e^{2x} = 12e^{2x}(e^x + 1)$

$h'(x) > 0, \forall x \in R$

$h(x)$ is an increasing function.

Minimum value of $h(x)$ will be when

$x \to -\infty$ at $[h(x)]_{min} = -1$ and $[h(x)]_{max} = \infty$

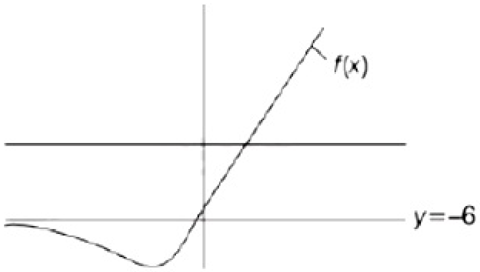
$f'(x) = g(x) \cdot h(x)$

Now, $h(x)$ is an increasing function and $h(x)$ varies from -1 to $+\infty$. So, this implies that $h(x)$ cuts the X -axis at one point and which further implies $f(x)$ changes its sign only at one point. Let's say at $x = \alpha$

$f(x) = e^{4x} + 2e^{3x} - e^x - 6$

When, $x \to -\infty$; $f(x) \to -6$

$x \to +\infty$; $f(x) \to +\infty$



So, $f(x)$ cuts the X-axis at a single point.

Question94

If R is the least value of a such that the function $f(x) = x^2 + ax + 1$ is increasing on [1, 2] and S is the greatest value of a such that the function $f(x) = x^2 + ax + 1$ is decreasing on [1, 2], then the value of $|R - S|$ is
[2021, 31 Aug. Shift-1]

Answer: 2

Solution:

Solution:
$f(x) = x^2 + ax + 1$
$f'(x) = 2x + a$
According to the question, $f'(x) \geq 0$ for $x \in [1, 2]$
For the least value $2x + a \geq 0$
$\Rightarrow a \geq -2x \Rightarrow a \geq -2 \Rightarrow R = -2$
For the greatest value $2x + a \leq 0$
$\Rightarrow a \leq -2x \; \{x \in [1, 2]\}$
$\Rightarrow a \leq -4$
$\Rightarrow S = -4$
$|R - S| = |-2 + 4| = 2$

Question95

Let f be any continuous function on [0, 2] and twice differentiable on (0, 2). If $f(0) = 0$, $f(1) = 1$ and $f(2) = 2$, then
[2021, 31 Aug. Shift-II]

Options:

A. $f''(x) = 0$ for all $x \in (0, 2)$

B. $f''(x) = 0$ for some $x \in (0, 2)$

C. $f'(x) = 0$ for some $x \in [0, 2]$

D. $f''(x) > 0$ for all $x \in (0, 2)$

Answer: B

Solution:

Solution:
$f(0) = 0$, $f(1) = 1$ and $f(2) = 2$
Let $h(x) = f(x) - x$
Clearly $h(x)$ is continuous and twice differentiable on $(0, 2)$
Also, $h(0) = h(1) = h(2) = 0$
$\therefore h(x)$ satisfies all the condition of Rolle's theorem.
$\therefore$ There exist $C_1 \in (0, 1)$ such that $h'(c_1) = 0$
$\Rightarrow f'(c_1) - 1 = 0$
$\Rightarrow f'(c_1) = 1$
also there exist $c_2 \in (1, 2)$ such that $h'(c_2) = 0$
$f'(c_2) = 1$
Now, using Rolle's theorem on $[c_1, c_2]$ for $f'(x)$
We have $f''(c) = 0, c \in (c_1, c_2)$
Hence, $f''(x) = 0$ for some $x \in (0, 2)$.

Question96

Let A be the set of all points (α, β) such that the area of triangle formed by the points (5, 6), (3, 2) and (α, β) is 12 sq units. Then, the least possible length of a line segment joining the origin to a point in A, is [31 Aug 2021 Shift 2]

Options:

A. $4\sqrt{5}$

B. $\frac{16}{\sqrt{5}}$

C. $\frac{8}{\sqrt{5}}$

D. $\frac{12}{\sqrt{5}}$

Answer: C

Solution:

Solution:
Area = 12 sq unitts

$$\Rightarrow \begin{vmatrix} \alpha & \beta & 1 \\ 5 & 6 & 1 \\ 3 & 2 & 1 \end{vmatrix} = \pm 24$$

$\Rightarrow 4\alpha - 2\beta - 8 = \pm 24$
$\Rightarrow 4\alpha - 2\beta = 32, 4\alpha - 2\beta + 16 = 0$
$\Rightarrow 2\alpha - \beta - 16 = 0, 2\alpha - \beta + 8 = 0$
Distance from origin when $\beta = 2\alpha + 8$ is
$D = \sqrt{\alpha^2 + (2\alpha + 8)^2} = \sqrt{5\alpha^2 + 32\alpha + 64}$

Now, $\frac{d}{d\alpha}(D^2) = 0$

$\Rightarrow 10\alpha + 32 = 0$

$\alpha = -\frac{16}{5}$

$\Rightarrow \beta = -\frac{32}{5} + 8 = \frac{8}{5}$

$\therefore D = \sqrt{\left(\frac{-16}{5}\right)^2 + \left(\frac{8}{5}\right)^2} = \frac{8}{5}\sqrt{5} = \frac{8}{\sqrt{5}}$

Similarly, if $\beta = 2\alpha - 16$

$D = \frac{16}{\sqrt{5}}$

So, least possible length of line segment $= \frac{8}{\sqrt{5}}$.

Question97

Let $f(x)$ be a cubic polynomial with $f(1) = -10$, $f(-1) = 6$, and has a local minima at $x = 1$, and $f'(x)$ has a local minima at $x = -1$. Then $f(3)$ is equal to.
[31 Aug 2021 Shift 2]

Answer: 22

Solution:

Solution:

let $f(x) = ax^3 + bx^2 + cx + d$

$\because$ and $(1) = -10$ and $f(-1) = 6$

$\therefore a + b + c + d = -10$

and $-a + b - c + d = 6$

$\because f(x)$ has a local minima at $x = 1$

$\therefore f'(1) = 0$

and $f'(x)$ has a local minima at $x = -1$

$\therefore f''(-1) = 0$

$\because f(x) = ax^3 + bx^2 + cx + d$

$\therefore f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$\because f''(-1) = 0$

$= -6a + 2b = 0$

$= b = 3a...(3)$

also $f'(1) = 0$

$= 3a + 2b + c = 0$

$= c = -9a$.

By adding (1) and (2), we get

$2b + 2d = -4$

$= b + d = -2$

$= 3a + d = -2$

$= d = -2 - 3a$.

Put $b = 3a$, $c = -9a$ and $d = -2 - 3a$ in (1) we get

$a + 3a - 9a - 2 - 3a = -10$

$= -8a = -10 + 2 = -8$

$a = \frac{-8}{-8} = 1$

$\therefore b = 3, c = -9$ and $d = -2 - 3 = -5$

$\therefore f(x) = x^3 + 3x^2 - 9x - 5$

$\therefore f(3) = 3^3 + 3.3^2 - 9 \times 3 - 5$

= 22

Question98

An angle of intersection of the curves, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $x^2 + y^2 = ab$, $a > b$, is
[31 Aug 2021 Shift 2]

Options:

A. $\tan^{-1}\left(\frac{a+b}{\sqrt{ab}}\right)$

B. $\tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$

C. $\tan^{-1}\left(\frac{a-b}{\sqrt{ab}}\right)$

D. $\tan^{-1}(2\sqrt{ab})$

Answer: C

Solution:

Solution:
Given curves
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i)
and $x^2 + y^2 = ab$ ...(ii)
From Eqs. (ii)
$y^2 = ab - x^2$
From Eq. (i),
$b^2x^2 + a^2(ab - x^2) = a^2b^2$
$(b^2 - a^2)x^2 = a^2b(b - a)$
$\Rightarrow x^2 = \frac{a^2b}{a+b}$
$y^2 = ab - \frac{a^2b}{a+b} = \frac{ab^2}{a+b}$
Point of intersection $\left(\sqrt{\frac{a^2b}{a+b}}, \sqrt{\frac{ab^2}{a+b}}\right)$
Now, differentiating Eq. (i) w.r.t. x, we have
$\frac{dy}{dx} = -\frac{b^2x}{a^2y} = m_1$ (Let)
and differentiating Eq. (ii) w.r.t. x,
$\frac{dy}{dx} = \frac{-x}{y} = m_2$ (Let)
Let angle be θ, Then,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\frac{-b^2x}{a^2y} + xy}{1 + \frac{b^2x^2}{a^2y^2}}\right|$$

$$= \left|\frac{xy(a^2 - b^2)}{a^2b^2}\right| = \left|\sqrt{\frac{a^3b^3}{(a+b)^2}} \cdot \frac{(a^2 - b^2)}{a^2b^2}\right|$$

$$= \left|\frac{a-b}{\sqrt{ab}}\right|$$

$\Rightarrow \theta = \tan^{-1}\left(\frac{a-b}{\sqrt{ab}}\right)$

Question99

A wire of length 20m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a regular hexagon. Then, the length of the side (in m) of the hexagon, so that the combined area of the square and the hexagon is minimum, is
[27 Aug 2021 Shift 1]

Options:

A. $\frac{5}{2+\sqrt{3}}$

B. $\frac{10}{2+3\sqrt{3}}$

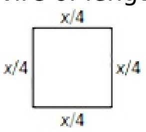
C. $\frac{5}{3+\sqrt{3}}$

D. $\frac{10}{3+2\sqrt{3}}$

Answer: D

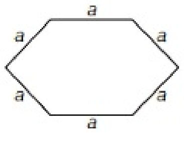
Solution:

Solution:
Let two pieces of wire one of length x and other of the length $20 - x$.
Wire of length x is made into a square



∴ Area of square $= \left(\frac{x}{4}\right)^2 = A_S(\text{let})$

Wire of length $(20 - x)$ is made into a regular hexagon.



Area of hexagon $= 6 \times \frac{\sqrt{3}}{4}a^2$

$(\text{Let})A_H = \frac{3\sqrt{3}}{2}\left(\frac{20-x}{6}\right)^2 \left[\because a = \frac{20-x}{6}\right]$

Sum of both area

$A = A_S + A_H$

$= \frac{x^2}{16} + \frac{\sqrt{3}}{24}(20-x)^2$

$\frac{dA}{dx} = \frac{x}{8} - \frac{\sqrt{3}}{12}(20-x) = \frac{3x - 40\sqrt{3} + 2\sqrt{3}x}{24}$

$\frac{dA}{dx} = 0$

$\Rightarrow x = \frac{40\sqrt{3}}{3+2\sqrt{3}} = \frac{40}{\sqrt{3}+2} = 40(2-\sqrt{3})$

$\frac{d^2A}{dx^2} = \frac{3+2\sqrt{3}}{24} > 0$

$\therefore$ Side of hexagon $= \frac{20 - 40(2 - \sqrt{3})}{6}$

$= \frac{20\sqrt{3} - 30}{3}$

$= \frac{10(2\sqrt{3} - 3)}{3} = \frac{10}{2\sqrt{3} + 3}$

Question100

A box open from top is made from a rectangular sheet of dimension a × b by cutting squares each of side x from each of the four corners and folding up the flaps. If the volume of the box is maximum, then x is equal to
[27 Aug 2021 Shift 2]

Options:

A. $\frac{a + b - \sqrt{a^2 + b^2 - ab}}{12}$

B. $\frac{a + b - \sqrt{a^2 + b^2 + ab}}{6}$

C. $\frac{a + b - \sqrt{a^2 + b^2 - ab}}{6}$

D. $\frac{a + b + \sqrt{a^2 + b^2 - ab}}{6}$

Answer: C

Solution:

Solution:
Length of box = a − 2x
Breadth of box = b − 2x
Height of box = x
Volume of box, V = (a − 2x) (b − 2x)x
$\Rightarrow V = 4x^3 - 20x^2 - 2bx^2 + abx$
Differentiating V w.r.t. x,
$V'_{(x)} = 12x^2 - 4(a + b)x + ab$
Critical Point, $V'_{(x)} = 0$
$\Rightarrow 12x^2 - 4(a + b)x + ab = 0$
$\Rightarrow x = \frac{4(a + b) \pm \sqrt{16(a + b)^2 - 4.12 . ab}}{2(12)}$
$x = \frac{(a + b) \pm \sqrt{a^2 + b^2 - ab}}{6}$
$V''(x) = 24x - 4(a + b)$
For $x = \frac{(a + b) - \sqrt{a^2 + b^2 - ab}}{6}$, $V''(x) < 0$
Hence, for maximum volume
$x = \frac{a + b - \sqrt{a^2 + b^2 - ab}}{6}$

Question101

A wire of length 36m is cut into two pieces, one of the pieces is bent to form a square and the other is bent to form a circle. If the sum of the areas of the two figures is minimum and the circumference of the circle is k(m), then $\left(\frac{4}{\pi}+1\right)$k is equal to

[26 Aug 2021 Shift 1]

Answer: 36

Solution:

Solution:

Let x + y = 36

where, x is perimeter of square and y is perimeter of circle

Then, side of square $=\frac{x}{4}$ and radius of circle $=\frac{y}{2\pi}$

Now, Sum of areas of square and circle, $A=\frac{x^2}{16}+\frac{y^2}{4\pi}$

$\Rightarrow A=\frac{x^2}{16}+\frac{(36-x)^2}{4\pi}\ [\because y=36-x]$

For minimum area

$\frac{dA}{dx}=0$

Now, $\frac{dA}{dx}=\frac{2x}{16}+\frac{-2(36-x)}{4\pi}=0$

$\Rightarrow x=\frac{144}{\pi+4}$

Circumference of circle = y

$=(36-x)$

$=36-\frac{144}{\pi+4}=\frac{36\pi}{\pi+4}$

According to the question,

$k=\frac{36\pi}{\pi+4}$

$\Rightarrow\left(\frac{4}{\pi}+1\right)k=\left(\frac{4}{\pi}+1\right)\frac{36\pi}{\pi+4}=36$

Question102

The local maximum value of the function $f(x)=\left(\frac{2}{x}\right)^{x^2}$, x > 0

[26 Aug 2021 Shift 2]

Options:

A. $(2\sqrt{e})^{\frac{1}{e}}$

B. $\left(\frac{4}{\sqrt{e}}\right)^{\frac{e}{4}}$

C. $(e)^{\frac{2}{e}}$

D. 1

Answer: C

Solution:

Solution:

$f(x) = \left(\frac{2}{x}\right)^{x^2}; x > 0$

$\therefore \log f(x) = x^2(\log 2 - \log x)$

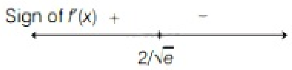
$f'(x) = f(x)[-x + (\log 2 - \log x)2x]$

$f'(x) = f(x) \,.\, x(2\log 2 - 2\log x - 1)$

For maxima or minima put $f'(x) = 0$, we get

$2\log 2 - 2\log x - 1 = \log\left(\frac{4}{x^2}\right) - 1 = 0$

$\Rightarrow x = \frac{2}{\sqrt{e}}$



$\therefore$ At $x = \frac{2}{\sqrt{e}}$, $f(x)$ has maximum value.

Maximum $= \left(\frac{2}{2\sqrt{e}}\right)^{\frac{4}{e}} = e^{\frac{2}{e}}$

Question103

**The function $f(x) = x^3 - 6x^2 + ax + b$ is such that $f(2) = f(4) = 0$. Consider two statements.
(S_1) there exists $x_1, x_2 \in (2, 4)$, $x_1 < x_2$, such that $f'(x_1) = -1$ and $f'(x_2) = 0$. (S_2) there exists $x_3, x_4 \in (2, 4)$, $x_3 < x_4$, such that f is decreasing in $(2, x_4)$, increasing in $(x_4, 4)$ and $2f'(x_3) = \sqrt{3}f(x_4)$. Then, [2021, 01 Sep. Shift-II]**

Options:

A. both (S_1) and (S_2) are true

B. (S_1) is false and (S_2) is true

C. both (S_1) and (S_2) are false

D. (S_1) is true and (S_2) is false

Answer: A

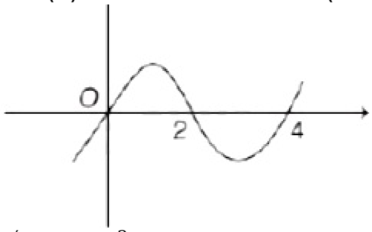
Solution:

Solution:

$f(x) = x^3 - 6x^2 + ax + b$

On solving a = 8, b = 0

$\therefore\ f(x) = x^3 - 6x^2 + 8x = x(x-2)(x-4)$



$f'(x) = 3x^2 - 12x + 8$

$f'(x) = 0 \Rightarrow x = 2 \pm \frac{2}{\sqrt{3}}$

$\because\ f'(x_2) = 0$ and $x_2 \in (2, 4)$

$\Rightarrow\ x_2 = 2 + \frac{2}{\sqrt{3}}$ and $f'(x_1) = -1$

$\Rightarrow\ 3x_1^2 - 12x_1 + 8 = 0$

$\Rightarrow\ x_1 = 1, 3$ (S_1 is true)

Now, $2(3x^2 - 12x + 8)$

$= \sqrt{3}\left(2 + \frac{2}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}} - 2\right)$

$\Rightarrow\ x = \frac{8}{3}, \frac{4}{3}$ (S_2 is true)

Question104

Let $f(x) = x\cos^{-1}(-\sin|x|)$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then which of the following is true?
[Jan. 8, 2020 (I)]

Options:

A. f ' is increasing in $\left(-\frac{\pi}{2}, 0\right)$ and decreasing in $\left(0, \frac{\pi}{2}\right)$

B. $f'(0) = -\frac{\pi}{2}$

C. f 'is not differentiable at x = 0

D. f ' is decreasing in $\left(-\frac{\pi}{2}, 0\right)$ and increasing in $\left(0, \frac{\pi}{2}\right)$

Answer: D

Solution:

Solution:

$'(x) = x(\pi - \cos^{-1}(\sin|x|))$

$= x\left(\pi - \left(\frac{\pi}{2} - \sin^{-1}(\sin|x|)\right)\right) = x\left(\frac{\pi}{2} + |x|\right)$

$$f(x) = \begin{cases} x\left(\frac{\pi}{2} + x\right), & x \ge 0 \\ x\left(\frac{\pi}{2} - x\right), & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{\pi}{2} + 2x, & x \geq 0 \\ \frac{\pi}{2} - 2x, & x < 0 \end{cases}$$

Hence, $f'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$ and decreasing in $\left(\frac{-\pi}{2}, 0\right)$

Question105

Let f be any function continuous on [a, b] and twice differentiable on (a, b). If for all $x \in (a, b)$, $f2(x) > 0$ and $f''(x) < 0$, then for any $c \in (a, b)$, $\frac{f(c) - f(a)}{f(b) - f(c)}$ is greater than:
[Jan. 9, 2020 (I)]

Options:

A. $\frac{b + a}{b - a}$

B. 1

C. $\frac{b - c}{c - a}$

D. $\frac{c - a}{b - c}$

Answer: D

Solution:

Solution:
Since, function $f(x)$ is twice differentiable and continuous in $x \in [a, b]$. Then, by LMVT for $x \in [a, c] \frac{f(c) - f(a)}{c - a} = f'(\alpha), \alpha \in (a, c)$
Again by LMVT for $x \in [c, b]$
$\frac{f(b) - f(c)}{b - c} = f'(\beta), \beta \in (c, b)$
$\because f''(x) < 0 \Rightarrow f'(x)$ is decreasing
$f'(\alpha) > f'(\beta) \Rightarrow \frac{f(c) - f(a)}{c - a} > \frac{f(b) - f(c)}{b - c}$
$\Rightarrow \frac{f(c) - f(a)}{f(b) - f(c)} > \frac{c - a}{b - c}$ ($\because f(x)$ is increasing)

Question106

A spherical iron ball of 10cm radius is coated with a layer of ice of uniform thickness that melts at a rate of $50 cm^3$/ min. When the thickness of ice is 5cm, then the rate (in cm / min.) at which of the thickness of ice decreases, is:
[Jan. 9,2020 (I)]

Options:

A. $\frac{5}{6\pi}$

B. $\frac{1}{54\pi}$

C. $\frac{1}{36\pi}$

D. $\frac{1}{18\pi}$

Answer: D

Solution:

Solution:
Let the thickness of ice layer be $= x cm$
Total volume $V = \frac{4}{3}\pi(10+x)^3$
$\frac{dV}{dt} = 4\pi(10+x)^2\frac{dx}{dt}$ .....(i)
Since, it is given that
$\frac{dV}{dt} = 50 cm^3 / min$ ......(ii)
From (i) and (ii), $50 = 4\pi(10+x)$
$\Rightarrow 50 = 4\pi(10+5)^2\frac{dx}{dt}$[$\because$ thickness of ice $x = 5$]
$\Rightarrow \frac{dx}{dt} = \frac{1}{18\pi} cm / min$

Question107

Let the normal at a point P on the curve $y^2 - 3x^2 + y + 10 = 0$ intersect the y -axis at (0, 32). If m is the slope of the tangent at P to the curve, then |m| is equal to ______.
[NA Jan. 8, 2020 (I)]

Answer: 4

Solution:

Solution:
$P \equiv (x_1, y_1)$
$2yy' - 6x + y' = 0$
$\Rightarrow y' = \left(\frac{6x_1}{1+2y_1}\right)$
$\Rightarrow \left(\frac{\frac{3}{2} - y_1}{-x_1}\right) = -\left(\frac{1+2y_1}{6x_1}\right)$ [By point slope form, $y - y_1 = m(x - x_1)$]
$\Rightarrow 9 - 6y_1 = 1 + 2y_1$
$\Rightarrow y_1 = 1$
$\therefore x_1 = \pm 2$
$\therefore$ Slope of tangent $(m) = \left(\frac{\pm 12}{3}\right) = \pm 4$

$\therefore |m| = 4$

Question108

The length of the perpendicular from the origin, on the normal to the curve, $x^2 + 2xy - 3y^2 = 0$ at the point (2,2) is:
[Jan. 8, 2020 (II)]

Options:

A. $\sqrt{2}$

B. $4\sqrt{2}$

C. 2

D. $2\sqrt{2}$

Answer: D

Solution:

Solution:
Given equation of curve is $x^2 + 2xy - 3y^2 = 0$
$\Rightarrow 2x + 2y + 2xy' - 6yy' = 0$
$\Rightarrow x + y + xy' - 3yy' = 0$
$\Rightarrow y'(x - 3y) = -(x + y)$
$\Rightarrow \frac{dy}{dx} = \frac{x + y}{3y - x}$
Slope of normal $= \frac{-dx}{dy} = \frac{x - 3y}{x - 3y}$
Normal at point $(2, 2) = \frac{2 - 6}{2 + 2} = -1$
Equation of normal to curve $= y - 2 = -1(x - 2)$
$\Rightarrow x + y = 4$
$\therefore$ Perpendicular distance from origin
$= \left| \frac{0 + 0 - 4}{\sqrt{2}} \right| = 2\sqrt{2}$

Question109

Let $f(x)$ be a polynomial of degree 3 such that $f(-1) = 10$, $f(1) = -6$, $f(x)$ has a critical point at $x = -1$ and $f'(x)$ has a critical point at $x = 1$. Then $f(x)$ has a local minima at $x =$ _______.
[NA Jan. 8, 2020 (II)]

Answer: 3

Solution:

Let $f(x) = ax^3 + bx^2 + cx + d$
$f(-1) = 10$ and $f(1) = -6$
$-a + b - c + d = 10$ ....(i)
$a + b + c + d = -6$ ....(ii)
Solving equations (i) and (ii), we get
$a = \frac{1}{4}, \ d = \frac{35}{4}$
$b = \frac{-3}{4}, \ c = -\frac{9}{4}$
$\Rightarrow f(x) = a(x^3 - 3x^2 - 9x) + d$
$f'(x) = \frac{3}{4}(x^2 - 2x - 3) = 0$
$\Rightarrow x = 3, -1$

Local minima exist at x = 3

Question110

Let f(x) be a polynomial of degree 5 such that x = ±1 are its critical points. If = 4, then which one of the following is not true ? [Jan. 7, 2020 (II)]

Options:

A. f is an odd function.

B. f(1) – 4f(–1) = 4.

C. x = 1 is a point of maxima and x = –1 is a point ofminima of f.

D. x = 1 is a point of minima and x = –1 is a point ofmaxima of f

Answer: D

Solution:

Solution:
$f(x) = ax^5 + bx^4 + cx^3$
$\lim_{x \to 0} \left(2 + \frac{ax^5 + bx^4 + cx^3}{x^3}\right) = 4$
$\Rightarrow 2 + c = 4 \Rightarrow c = 2$
$f'(x) = 5ax^4 + 4bx^3 + 6x^2$
$= x^2(5ax^2 + 4bx + 6)$
Since, $x = \pm 1$ are the critical points,
$\therefore f'(1) = 0 \Rightarrow 5a + 4b + 6 = 0$ ...(i)
$f'(-1) = 0 \Rightarrow 5a - 4b + 6 = 0$ ...(ii)
From eqns. (i) and (ii),
$b = 0$ and $a = -\frac{6}{5}$
$f(x) = \frac{-6}{5}x^5 + 2x^3$
$f'(x) = -6x^4 + 6x^2 = 6x^2(-x^2 + 1)$
$= -6x^2(x + 1)(x - 1)$

$\therefore f(x)$ has minima at $x = -1$ and maxima at $x = 1$

Question111

Let m and M be respectively the minimum and maximum values of

$$\begin{vmatrix} \cos^2 x & 1+\sin^2 x & \sin 2x \\ 1+\cos^2 x & \sin^2 x & \sin 2x \\ \cos^2 x & \sin^2 x & 1+\sin 2x \end{vmatrix}$$

Then the ordered pair (m, M) is equal to :
[Sep. 06, 2020 (I)]

Options:

A. (-3,3)

B. (-3,-1)

C. (-4,-1)

D. (1,3)

Answer: B

Solution:

Solution:
$C_1 \to C_1 + C_2$
Let

$$f(x) = \begin{vmatrix} 2 & 1+\sin^2 x & \sin 2x \\ 2 & \sin^2 x & \sin 2x \\ 1 & \sin^2 x & 1+\sin 2x \end{vmatrix}$$

$R_1 \to R_1 - 2R_3;\ R_2 \to R_2 - 2R_3$

$$= \begin{vmatrix} 0 & \cos^2\theta & -(2+\sin 2x) \\ 0 & -\sin^2 x & -(2+\sin 2x) \\ 1 & \sin^2 x & 1+\sin 2x \end{vmatrix} = -2 - 2\sin 2x$$

$f'(x) = -2\cos 2x = 0$

$\Rightarrow \cos 2x = 0 \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$

$f''(x) = 4\sin 2x$

So, $f''\left(\frac{\pi}{4}\right) = 4 > 0$ (minima)

$m = f\left(\frac{\pi}{4}\right) = -2 - 1 = -3$

$f''\left(\frac{3\pi}{4}\right) = -4 < 0$ (maxima)

$M = f\left(\frac{3\pi}{4}\right) = -2 + 1 = -1$

So, $(m, M) = (-3, -1)$

Question112

The position of a moving car at time t is given by
$f(t) = at^2 + bt + c,\ t > 0$, where a, b and c are real numbers greater than

attained at the point :
[Sep.06, 2020 (I)]

Options:

A. $(t_2 - t_1) / 2$

B. $a(t_2 - t_1) + b$

C. $(t_1 + t_2) / 2$

D. $2a(t_1 + t_2) + b$

Answer: C

Solution:

Solution:

Average speed $= f'(t) = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$

$2at + b = a(t_1 + t_2) + b \Rightarrow t = \frac{t_1 + t_2}{2}$

Question113

If the surface area of a cube is increasing at a rate of 3.6 cm^2/sec, retaining its shape; then the rate of change of its volume (in cm^3 /sec.), when the length of a side of the cube is 10 cm, is :
[Sep. 03, 2020 (II)]

Options:

A. 18

B. 10

C. 20

D. 9

Answer: D

Solution:

Solution:

Let the side of cube be a.

$S = 6a^2 \Rightarrow \frac{dS}{dt} = 12a \cdot \frac{da}{dt} \Rightarrow 3.6 = 12a \cdot \frac{da}{dt}$

$\Rightarrow 12(10)\frac{da}{dt} = 3.6 \Rightarrow \frac{da}{dt} = 0.03$

$V = a^3 \Rightarrow \frac{dV}{dt} = 3a^2 \cdot \frac{da}{dt} = 3(10)^2 \cdot \left(\frac{3}{100}\right) = 9$

Question114

If a function f (x) defined by

$$f(x) = \begin{cases} ae^x + be^{-x} & , \ -1 \le x < 1 \\ cx^2 & , \ 1 \le x \le 3 \\ ax^2 + 2cx & , \ 3 < x \le 4 \end{cases}$$

be continuous for some a, b, c ∈ R and $f'(0) + f'(2) = e$, then the value of a is :
[Sep. 02, 2020 (I)]

Options:

A. $\frac{1}{e^2 - 3e + 13}$

B. $\frac{e}{e^2 - 3e - 13}$

C. $\frac{e}{e^2 + 3e + 13}$

D. $\frac{e}{e^2 - 3e + 13}$

Answer: D

Solution:

Solution:
Since, function f (x) is continuous at x = 1, 3
$\therefore f(1) = f(1^+)$
$\Rightarrow ae + be^{-1} = c$ ......(i)
$f(3) = f(3^+)$
$\Rightarrow 9c = 9a + 6c \Rightarrow c = 3a$ ....(ii)
From (i) and (ii),
$b = ae(3 - e)$ ....(iii)

$$f'(x) = \begin{cases} ae^x - be^{-x} & -1 < x < 1 \\ 2cx & 1 < x < 3 \\ 2ax + 2c & 3 < x < 4 \end{cases}$$

$f'(0) = a - b, f'(2) = 4c$
Given, $f'(0) + f'(2) = e$
$a - b + 4c = e$ ......(iv)
From eqs. (i), (ii), (iii) and (iv),
$a - 3ae + ae^2 + 12a = e$
$\Rightarrow 13a - 3ae + ae^2 = e$
$\Rightarrow a = \frac{e}{e^2 - 3e + 13}$

Question115

$f(x) = (3x - 7)x^{2/3}$, x ∈ R, is increasing for all x lying in :
[Sep. 03, 2020 (I)]

A. $(-\infty, 0) \cup \left(\frac{14}{15}, \infty\right)$

B. $(-\infty, 0) \cup \left(\frac{3}{7}, \infty\right)$

C. $\left(-\infty, \frac{14}{15}\right)$

D. $\left(-\infty, -\frac{14}{15}\right) \cup (0, \infty)$

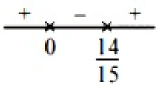
Answer: A

Solution:

Solution:
$f(x) = (3x - 7) . x^{2/3}$

$f'(x) = 3x^{2/3} + (3x - 7) . \frac{2}{3}x^{-\frac{1}{3}}$

$= \frac{15x - 14}{3x^{1/3}}$



For increasing function

$f'(x) > 0$ then $x \in (-\infty, 0) \cup \left(\frac{14}{15}, \infty\right)$

Question116

If the tangent to the curve, $y = f(x) = x\log_e x$, $(x > 0)$ at a point $(c, f(c))$ is parallel to the line segement joining the points (1,0) and (e, e), then c is equal to:
[Sep. 06, 2020 (II)]

Options:

A. $\frac{e - 1}{e}$

B. $e^{\left(\frac{1}{e - 1}\right)}$

C. $e^{\left(\frac{1}{1 - e}\right)}$

D. $\frac{e}{e - 1}$

Answer: B

Solution:

Solution:
The given tangent to the curve is

$\Rightarrow \frac{dy}{dx} = 1 + \log_e x$

$\Rightarrow \left.\frac{dy}{dx}\right]_{x=c} = 1 + \log_e c$ (slope)

∵ The tangent is parallel to line joining (1, 0), (e, e)

$\therefore 1 + \log_e c = \frac{e-0}{e-1}$

$\Rightarrow \log_e c = \frac{e}{e-1} - 1 \Rightarrow \log_e c = \frac{1}{e-1}$

$\Rightarrow c = e^{\frac{1}{e-1}}$

Question117

Which of the following points lies on the tangent to the curve $x^4 e^y + 2\sqrt{y+1} = 3$ at the point (1,0) ?
[Sep. 05, 2020 (II)]

Options:

A. (2, 2)

B. (2, 6)

C. (– 2, 6)

D. (– 2, 4)

Answer: C

Solution:

Solution:
The given curve is, $x^4 . e^y + 2\sqrt{y+1} = 3$
Differentiating w.r.t. x, we get

$(4x^3 + x^4 . y')e^y + \frac{y'}{\sqrt{1+y}} = 0$

$\Rightarrow \left(\frac{dy}{dx}\right) = \frac{-4x^3 e^y}{\left(\frac{1}{\sqrt{y+1}} + e^y x^4\right)}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = -2$

∴ Equation of tangent;
$y - 0 = -2(x-1) \Rightarrow 2x + y = 2$
Only point (-2,6) lies on the tangent.

Question118

If the lines $x + y = a$ and $x - y = b$ touch the curve $y = x^2 - 3x + 2$ at the points where the curve intersects thex -axis, then $\frac{a}{b}$ is equal to _______.
[NA Sep. 05,2020 (II)]

Answer: 0.50

Solution:

Solution:
The given curve $y = (x-1)(x-2)$, intersects the x -axis at $A(1, 0)$ and $B(2, 0)$
$\therefore \frac{dy}{dx} = 2x - 3;\ \left(\frac{dy}{dx}\right)_{(x=1)} = -1$ and $\left(\frac{dy}{dx}\right)_{(x=2)} = 1$
Equation of tangent at $A(1, 0)$
$y = -1(x-1) \Rightarrow x + y = 1$
Equation of tangent at $B(2, 0)$
$y = 1(x-2) \Rightarrow x - y = 2$
So $a = 1$ and $b = 2$
$\Rightarrow \frac{a}{b} = \frac{1}{2} = 0.5$

Question119

If the tangent to the curve, $y = e^x$ at a point (c, e^c) and the normal to the parabola, $y^2 = 4x$ at the point (1,2) intersect at the same point on the x - axis, then the value of c is ________.
[NA Sep. 03,2020 (II)]

Answer: 4

Solution:

For (1,2) of $y^2 = 4x \Rightarrow t = 1, a = 1$
Equation of normal to the parabola
$\Rightarrow tx + y = 2at + at^3$
$\Rightarrow x + y = 3$ intersect x -axis at (3,0)
$y = e^x \Rightarrow \frac{dy}{dx} = e^x$
Equation of tangent to the curve
$\Rightarrow y - e^c = e^c(x - c)$
$\because$ Tangent to the curve and normal to the parabola intersect at same point.
$\therefore 0 - e^c = e^c(3 - c) \Rightarrow c = 4$

Question120

If $y = \sum_{k=1}^{6} k\cos^{-1}\left\{\frac{3}{5}\cos kx - \frac{4}{5}\sin kx\right\}$, then $\frac{dy}{dx}$ at $x = 0$ is ________.
[NA Sep. 02, 2020 (II)]

Answer: 91

Solution:

Solution:

$y=\sum_{k=1}^{6} k\cos^{-1}\left\{\frac{3}{5}\cos kx-\frac{4}{5}\sin kx\right\}$

Let $\cos a=\frac{3}{5}$ and $\sin a=\frac{4}{5}$

$\therefore y=\sum_{k=1}^{6} k\cos^{-1}\{\cos a\cos kx-\sin a\sin kx\}$

$=\sum_{k=1}^{6} k\cos^{-1}(\cos(kx+a))$

$=\sum_{k=1}^{6} k(kx+a)=\sum_{k=1}^{6}(k^2x+ak)$

$\therefore \frac{dy}{dx}=\sum_{k=1}^{6} k^2=\frac{6(7)(13)}{6}=91$

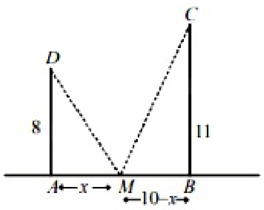
Question121

Let AD and BC be two vertical poles at A and B respectively on a horizontal ground. If AD = 8m, BC = 11m and AB = 10m; then the distance (in meters) of a point M on AB from the point A such that MD^2+MC^2 is minimum is
[NA Sep. 06, 2020 (I)]

Answer: 5

Solution:

Solution:



Let AM = x m

$\therefore (MD)^2+(MC)^2=64+x^2+121+(10-x)^2=f(x)$ (say)

$f'(x)=2x-2(10-x)=0$

$\Rightarrow 4x=20\Rightarrow x=5$

$f''(x)=2-2(-1)>0$

$\therefore f(x)$ is minimum at $x=5$m

Question122

The set of all real values of lambda for which the function

$f(x)=(1-\cos^2x).(\lambda+\sin x),\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, has exactly one maxima and

[Sep. 06, 2020 (II)]

Options:

A. $\left(-\frac{1}{2}, \frac{1}{2}\right) - \{0\}$

B. $\left(-\frac{3}{2}, \frac{3}{2}\right)$

C. $\left(-\frac{1}{2}, \frac{1}{2}\right)$

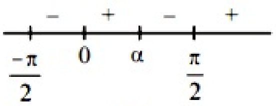
D. $\left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$

Answer: D

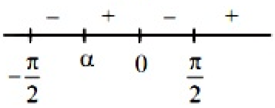
Solution:

Solution:
$f(x) = (1 - \cos^2 x)(\lambda + \sin x) = \sin^2 x(\lambda + \sin x)$
$\Rightarrow f(x) = \lambda \sin^2 x + \sin^3 x$ ......(i)
$\Rightarrow f'(x) = \sin x \cos x[2\lambda + 3\sin x] = 0$
$\Rightarrow \sin x = 0$ and $\sin x = -\frac{2\lambda}{3} \Rightarrow x = \alpha$ (let)

So, $f(x)$ will change its sign at $x = 0, \alpha$ because there is exactly one maxima and one minima in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$



OR



Now, $\sin x = -\frac{2\lambda}{3}$
$\Rightarrow -1 \le -\frac{2\lambda}{3} \le 1 \Rightarrow -\frac{3}{2} \le \lambda \le \frac{3}{2} - \{0\}$
$\because$ If $\lambda = 0 \Rightarrow f(x) = \sin^3 x$ (from (i))
Which is monotonic, then no maxima/minima
So, $\lambda \in \left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$

Question123

If $x = 1$ is a critical point of the function $f(x) = (3x^2 + ax - 2 - a)e^x$, then :

[Sep. 05, 2020 (II)]

Options:

A. $x = 1$ and $x = -\frac{2}{3}$ are local minima of f.

B. $x = 1$ and $x = -\frac{2}{3}$ are local maxima of f.

D. $x = 1$ is a local minima and $x = -\frac{2}{3}$ is a local maxima of f .

Answer: D

Solution:

Solution:
The given function
$f(x) = (3x^2 + ax - 2 - a)e^x$
$f'(x) = (6x + a)e^x + (3x^2 + ax - 2 - a)e^x$
$f'(x) = [3x^2 + (a + 6)x - 2]e^x$
$\because x = 1$ is critical point :
$\therefore f'(1) = 0$
$\Rightarrow (3 + a + 6 - 2) . e = 0$
$\Rightarrow a = -7 \quad (\because e > 0)$
$\therefore f'(x) = (3x^2 - x - 2)e^x$
$= (3x + 2)(x - 1)e^x$

$\therefore x = -\frac{2}{3}$ is point of local maxima.
and $x = 1$ is point of local minima.

Question124

The area (in sq. units) of the largest rectangle ABCD whose vertices A and B lie on the x -axis and vertices C and D lie on the parabola, $y = x^2 - 1$ below the x -axis, is: [Sep. 04, 2020 (II)]

Options:

A. $\frac{2}{3\sqrt{3}}$

B. $\frac{1}{3\sqrt{3}}$

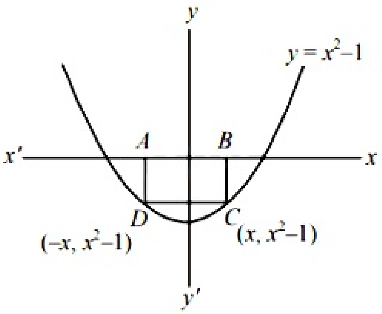
C. $\frac{4}{3}$

D. $\frac{4}{3\sqrt{3}}$

Answer: D

Solution:

Solution:
Area of rectangle ABCD
$A = 2x . (x^2 - 1) = 2x^3 - 2x$
$\therefore \frac{dA}{dx} = 6x^2 - 2$
For maximum area $\frac{dA}{dx} = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$
$\frac{d^2A}{dx^2} = (12x) \Rightarrow \left(\frac{d^2A}{dx^2}\right)_{x = \frac{-1}{\sqrt{3}}} = \frac{-12}{\sqrt{3}} < 0$



∴ Maximum area $= \left|\frac{2}{3\sqrt{3}} - \frac{2}{\sqrt{3}}\right| = \frac{4}{3\sqrt{3}}$

Question125

Suppose f (x) is a polynomial of degree four, having critical points at −1, 0, 1 . If T = {x ∈ R | f (x) = f (0)}, then the sum of squares of all the elements of T is:
[Sep. 03, 2020 (II)]

Options:

A. 4

B. 6

C. 2

D. 8

Answer: A

Solution:

Solution:
∵ The critical points are -1,0,1
$\therefore f'(x) = k.x(x+1)(x-1) = k(x^3 - x)$
$\Rightarrow f(x) = k\left(\frac{x^4}{4} - \frac{x^2}{2}\right) + C$
$\Rightarrow f(0) = C$
$\because f(x) = f(0)$
$\Rightarrow k\frac{(x^4 - 2x^2)}{4} + C = C$
$\Rightarrow x^2(x^2 - 2) = 0$
$\Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$
$\Rightarrow T = \{0, \sqrt{2}, -\sqrt{2}\}$

Question126

If the function f given by $f(x) = x^3 - 3(a-2)x^2 + 3ax + 7$, for some a ∈ R is increasing in (0,1] and decreasing in [1, 5), then a root of the equation, $\frac{f(x) - 14}{(x-1)^2} = 0 (x \neq 1)$ is
[Jan. 12, 2019 (II)]

A. –7

B. 5

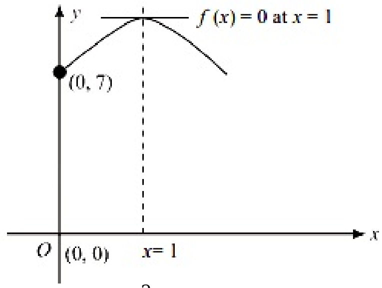
C. 7

D. 6

Answer: C

Solution:

Solution:
$f(x) = x^3 - 3(a-2)x^2 + 3ax + 7, f(0) = 7$



$\Rightarrow f'(x) = 3x^2 - 6(a-2)x + 3a$
$f'(1) = 0$
$\Rightarrow 1 - 2a + 4 + a = 0$
$\Rightarrow a = 5$
Then, $f(x) = x^3 - 9x^2 + 15x + 7$
Now,
$\frac{f(x) - 14}{(x-1)^2} = 0$
$\Rightarrow \frac{x^3 - 9x^2 + 15x + 7 - 14}{(x-1)^2} = 0$
$\Rightarrow \frac{(x-1)^2(x-7)}{(x-1)^2} = 0 \Rightarrow x = 7$

Question127

Let $f(x) = \frac{x}{\sqrt{a^2 + x^2}} - \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$, $x \in R$ where a, b and d are non-zero real constants.
[Jan. 11, 2019 (II)]

Options:

A. f is an increasing function of x

B. f is a decreasing function of x

C. f' is not a continuous function of x

D. f is neither increasing nor decreasing function of x

Answer: A

Solution:

$$f(x) = \frac{x}{\sqrt{a^2 + x^2}} - \frac{(d - x)}{\sqrt{b^2 + (d - x)^2}}$$
$$= \frac{x}{\sqrt{a^2 + x^2}} + \frac{(x - d)}{\sqrt{b^2 + (x - d)^2}}$$
$$f'(x) = \frac{\sqrt{a^2 + x^2} - \frac{x(2x)}{2\sqrt{a^2 + x^2}}}{(a^2 + x^2)} + \frac{\sqrt{b^2 + (x - d)^2} - \frac{(x - d)2(x - d)}{2\sqrt{b^2 + (x - d)^2}}}{(b^2 + (x - d)^2)}$$
$$= \frac{a^2 + x^2 - x^2}{(a^2 + x^2)^{3/2}} + \frac{b^2 + (x - d)^2 - (x - d)^2}{(b^2 + (x - d)^2)^{3/2}}$$
$$= \frac{a^2}{(a^2 + x^2)^{3/2}} + \frac{b^2}{(b^2 + (x - d)^2)^{3/2}} > 0$$
$\Rightarrow f'(x) > 0$, ₹$x \in R$
$\Rightarrow f(x)$ is increasing function.
Hence, $f(x)$ is increasing function.

Question128

The maximum area (in sq. units) of a rectangle having its base on the x - axis and its other two vertices on the parabola, $y = 12 - x^2$ such that the rectangle lies inside the parabola, is:
[Jan. 12, 2019 (I)]

Options:

A. 36

B. $20\sqrt{2}$

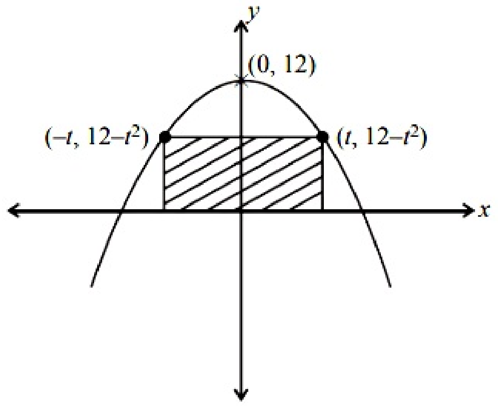
C. 32

D. $18\sqrt{3}$

Answer: C

Solution:

Solution:
Given, the equation of parabola is, $x^2 = 12 - y$



Area of the rectangle $= (2t)(12 - t^2)$
$A = 24t - 2t^3$
$$\frac{dA}{dt} = 24 - 6t^2$$
Put $\frac{dA}{dt} = 0 \Rightarrow 24 - 6t^2 = 0$
$\Rightarrow t = \pm 2$

Question129

The tangent to the curve $y = x^2 - 5x + 5$, parallel to the line $2y = 4x + 1$, also passes through the point:
[Jan. 12, 2019(II)]

Options:

A. $\left(\frac{7}{2}, \frac{1}{4}\right)$

B. $\left(\frac{1}{8}, -7\right)$

C. $\left(-\frac{1}{8}, 7\right)$

D. $\left(\frac{1}{4}, \frac{7}{2}\right)$

Answer: B

Solution:

Solution:
∵ Tangent to the given curve is parallel to line $2y = 4x + 1$
∴ Slope of tangent $(m) = 2$
Then, the equation of tangent will be of the form
$y = 2x + c$ ......(i)
∵ Line (i) and curve $y = x^2 - 5x + 5$ has only one point of intersection.
$\therefore 2x + c = x^2 - 5x + 5$
$x^2 - 7x + (5 - c) = 0$
$\therefore D = 49 - 4(5 - c) = 0$
$\Rightarrow c = -\frac{29}{4}$
Hence, the equation of tangent: $y = 2x - \frac{29}{4}$

Question130

The shortest distance between the point $\left(\frac{3}{2}, 0\right)$ and the curve $y = \sqrt{x}, (x > 0)$, is:
[Jan. 10, 2019 (I)]

Options:

A. $\frac{\sqrt{5}}{2}$

B. $\frac{\sqrt{3}}{2}$

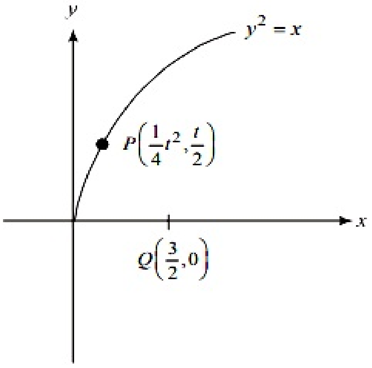
C. $\frac{3}{\ldots}$

D. $\frac{5}{4}$

Answer: A

Solution:

Solution:



Here the curve is parabola with $a = \frac{1}{4}$.

Let $P(at^2, 2at)$ or $P\left(\frac{t^2}{4}, \frac{t}{2}\right)$ be a point on the curve.

Now, $y^2 = x$

$\Rightarrow y\frac{dy}{dx} = 1 = \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{atp} = \frac{1}{t}$

$\therefore$ equation of normal at P to $y^2 = x$ is,

$\left(y - \frac{t}{2}\right) = -t\left(x - \frac{1}{4}t^2\right)$

$\Rightarrow y = -tx + \frac{1}{2}t + \frac{1}{4}t^3$ ....(i)

For minimum PQ, (i) passes through $Q\left(\frac{3}{2}, 0\right)$

$\frac{-3}{2}t + \frac{t}{2} + \frac{t^3}{4} = 0 \Rightarrow -4t + t^3 = 0$

$\Rightarrow t(t^2 - 4) = 0 \Rightarrow t = -2, 0, 2$

$\because t \geq 0 \Rightarrow t = 0, 2$

$t = 0, P(0, 0) \Rightarrow AP = \frac{3}{2}$

$t = 2, P(1, 1) \Rightarrow AP = \frac{\sqrt{5}}{2}$

Shortest distance $\left(\frac{3}{2}, 0\right)$ and $y = \sqrt{x}$ is $\frac{\sqrt{5}}{2}$

Question131

The tangent to the curve, $y = xe^{x^2}$ passing through the point (1, e) also passes through the point:
[Jan. 10,2019 (II)]

Options:

A. (2, 3e)

B. $\left(\frac{4}{3}, 2e\right)$

C. $\left(\frac{5}{3}, 2e\right)$

D. (3, 6e)

Answer: B

Solution:

Solution:
The equation of curve $y = xe^{x^2}$
$\Rightarrow \frac{dy}{dx} = e^{x^2}.1 + x.e^{x^2}.2x$
Since (1, e) lies on the curve $y = xe^{x^2}$, then equation of tangent at (1, e) is
$y - e = \left(e^{x^2}(1 + 2x^2)\right)_{x=1}(x - 1)$
$y - e = 3e(x - 1)$
$3ex - y = 2e$
So, equation of tangent to the curve passes through the point $\left(\frac{4}{3}, 2e\right)$

Question132

A helicopter is flying along the curve given by$y - x^{3/2} = 7$, $(x \geq 0)$. A soldier positioned at the point $\left(\frac{1}{2}, 7\right)$ wants to shoot down the helicopter when it is nearest to him. Then this nearest distance is:
[Jan. 10, 2019 (II)]

Options:

A. $\frac{\sqrt{5}}{6}$

B. $\frac{1}{3}\sqrt{\frac{7}{3}}$

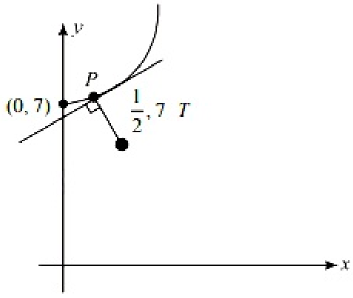
C. $\frac{1}{6}\sqrt{\frac{7}{3}}$

D. $\frac{1}{2}$

Answer: C

Solution:

Solution:
$f(x) = y = x^{3/2} + 7$
$\Rightarrow \frac{dy}{dx} \Rightarrow \frac{3}{2}\sqrt{x} > 0$
$\Rightarrow f(x)$ is increasing function $\forall x > 0$



Let P left $\left(x_1, x_1^{\frac{3}{2}} + 7\right)$

$m_{TP} = m_{\text{at } P} = -1$

$\Rightarrow \left(\frac{x_1^{3/2}}{x_1 - \frac{1}{2}}\right) \times \frac{3}{2} x_1^{\frac{1}{2}} = -1$

$\Rightarrow -\frac{2}{3} = \frac{x_1^{2}}{x_1 - \frac{1}{2}}$

$\Rightarrow -3x_1^{2} = 2x_1 - 1 \Rightarrow 3x_1^{2} + 2x_1 - 1 = 0$

$\Rightarrow 3x_1^{2} + 3x_1 - x_1 - 1 = 0$

$\Rightarrow 3x_1(x_1 + 1) - 1(x_1 + 1) = 0$

$\Rightarrow x_1 = \frac{1}{3}$

$\Rightarrow P\left(\frac{1}{3}, 7 + \frac{1}{3\sqrt{3}}\right)$

$TP = \sqrt{\frac{1}{27} + \frac{1}{36}} = \frac{1}{6}\sqrt{\frac{7}{3}}$

Question133

If θ denotes the acute angle between the curves, $y = 10 - x^2$ and $y = 2 + x^2$ at a point of their intersection, then $|\tan \theta|$ is equal to: [Jan. 09, 2019 (I)]

Options:

A. $\frac{4}{9}$

B. $\frac{8}{15}$

C. $\frac{7}{17}$

D. $\frac{8}{17}$

Answer: B

Solution:

Solution:

Since, the equation of curves are $y = 10 - x^2$

$y = 2 + x^2$ ......(i)

Adding eqn (i) and (ii), we get

$2y = 12 \Rightarrow y = 6$

Then, from eqn (i)

$\frac{dy}{dx} = -2x \Rightarrow \left(\frac{dy}{dx}\right)_{(2,6)} = -4$ and $= = \left(\frac{dy}{dx}\right)_{(-2,6)} = 4$

Differentiate equation (ii) with respect to x

$\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)_{(2,6)} = 4$ and $\left(\frac{dy}{dx}\right)_{(-2,6)} = -4$

At (2, 6) $\tan\theta = \left(\frac{(-4)-(4)}{1+(-4)\times(4)}\right) = \frac{8}{15}$

At (−2, 6), $\tan\theta = \frac{(4)-(-4)}{1+(4)(-4)} = \frac{8}{-15} \Rightarrow |\tan\theta| = \frac{8}{15}$

$\therefore |\tan\theta| = \frac{8}{15}$

Question134

The maximum value of $3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$ for any real value of θ is:
[Jan. 12, 2019 (I)]

Options:

A. $\sqrt{19}$

B. $\frac{\sqrt{79}}{2}$

C. $\sqrt{34}$

D. $\sqrt{31}$

Answer: A

Solution:

Let, the functions is,

$f(\theta) = 3\cos\theta + 5\sin\theta \,.\, \cos\frac{\pi}{6} - 5\sin\frac{\pi}{6}\cos\theta$

$= 3\cos\theta + 5 \times \frac{\sqrt{3}}{2}\sin\theta - 5\theta\frac{1}{2}\cos\theta$

$= \left(3 - \frac{5}{2}\right)\cos\theta + 5 \times \frac{\sqrt{3}}{2}\sin\theta$

$= \frac{1}{2}\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta$

$\max f(\theta) = \sqrt{\frac{1}{4} + \frac{25}{4} \times 3} = \sqrt{\frac{76}{4}} = \sqrt{19}$

Question135

Let P(4, −4) and Q(9, 6) be two points on the parabola, $y^2 = 4x$ and let this X be any point arc POQ of this parabola, where O is vertex of the parabola, such that the area of ΔPXQ is maximum. Then this minimum area (in sq. units) is:
[Jan. 12, 2019 (I)]

A. $\frac{75}{2}$

B. $\frac{125}{4}$

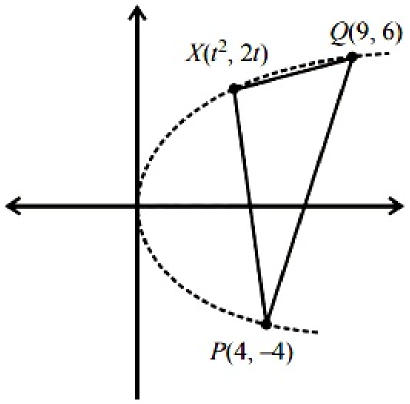
C. $\frac{625}{4}$

D. $\frac{125}{2}$

Answer: B

Solution:

Solution:



Parametric equations of the parabola $y^2 = 4x$ are,
$x = t^2$ and $y = 2t$

$$\text{Area } \Delta PXQ = \frac{1}{2}\begin{vmatrix} t^2 & 2t & 1 \\ 4 & -4 & 1 \\ 9 & 6 & 1 \end{vmatrix}$$

$= -5t^2 + 5t + 30$
$= -5(t^2 - t - 6)$
$= -5\left[\left(t - \frac{1}{2}\right)^2 - \frac{25}{4}\right]$

For maximum area $t = \frac{1}{2}$

$\therefore$ maximum area $= 5\left(\frac{25}{4}\right) = \frac{125}{4}$

Question136

The maximum value of the function $f(x) = 3x^3 - 18x^2 + 27x - 40$ on the set $S = \{x \in R : x^2 + 30 \leq 11x\}$ is
[Jan. 11, 2019 (I)]

Options:

A. -122

B. -222

C. 122

D. 222

Answer: C

Solution:

Solution:
Consider the function,
$f(x) = 3x(x - 3)^2 - 40$
Now $S = \{x \in \propto R : x^2 + 30 \leq 11x\}$
So $x^2 - 11x + 30 \leq 0 \Rightarrow x \propto \in [5, 6]$
$\therefore f(x)$ will have maximum value for $x = 6$
The maximum value of function is,
$f(6) = 3 \times 6 \times 3 \times 3 - 40 = 122$

Question137

Let x, y be positive real numbers and m, n positive integers. The maximum value of the expression $\frac{x^m y^n}{(1 + x^{2m})(1 + y^{2n})}$ is:
[Jan. 11, 2019 (II)]

Options:

A. 1

B. $\frac{1}{2}$

C. $\frac{1}{4}$

D. $\frac{m + n}{6mn}$

Answer: C

Solution:

Solution:
$A = \frac{x^m y^n}{(1 + x^{2m})(1 + y^{2n})} = \frac{1}{(x^{-m} + x^m)(y^{-n} + y^n)}$
$\frac{x^m + y^{-m}}{2} \geq (x^m . x^{-m})^{\frac{1}{2}} \Rightarrow x^m + x^m \geq 2$
In the same way, $y^n + y^n \geq 2$
Then, $(x^m + x^{-m})(y^{-n} + y^n) \geq 4$
$\Rightarrow \frac{1}{(x^m + x^{-m})(y^{-n} + y^n)} \leq \frac{1}{4}$

Question138

The maximum volume (in cu.m) of the right circular cone having slant height 3 m is:
[Jan. 09, 2019 (I)]

A. 6π

B. $3\sqrt{3}\pi$

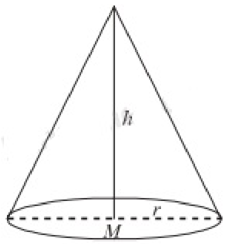
C. $\frac{4}{3}\pi$

D. $2\sqrt{3}\pi$

Answer: D

Solution:

Solution:



$h^2 + r^2 = l^2 = 9$......(i)

Volume of cone $V = \frac{1}{3}\pi r^2 h$ .....(ii)

From (i) and (ii),

$\Rightarrow V = \frac{1}{3}\pi(9 - h^2)h$

$\Rightarrow V = \frac{1}{3}\pi(9h - h^3) \Rightarrow \frac{dv}{dh} = \frac{1}{3}\pi(9 - 3h^2)$

For maxima/minima,

$\frac{dV}{dh} = 0 \Rightarrow \frac{1}{3}\pi(9 - 3h^2) = 0$

$\Rightarrow h = \pm\sqrt{3} \Rightarrow h = \sqrt{3} \because h > 0$

Now; $\frac{d^2V}{dh^2} = \frac{1}{3}\pi(-6h)$

Here, $\left(\frac{d^2V}{dh^2}\right)_{\text{at } h = \sqrt{3}} < 0$

Then, $h = \sqrt{3}$ is point of maxima

Hence, the required maximum volume is,

$V = \frac{1}{3}\pi(9 - 3)\sqrt{3} = 2\sqrt{3}\pi$

Question139

A spherical iron ball of radius 10 cm is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of the ice is 5 cm, then the rate at which the thickness (in cm/min) of the ice decreases, is :
[April 10, 2019 (II)]

Options:

A. $\frac{1}{18\pi}$

B. $\frac{1}{36\pi}$

D. $\frac{1}{9\pi}$

Answer: A

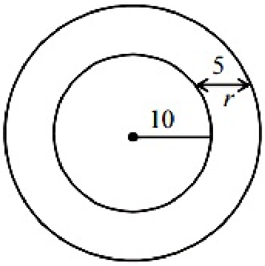
Solution:

Solution:
Given that ice melts at a rate of $50 cm^3 / min$.

$\therefore \frac{dV_{ice}}{dt} = 50$

$V_{ice} = \frac{4}{3}\pi(10 + r)^3 - \frac{4}{3}\pi(10)^3$

$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi 3(10 + r)^2 \frac{dr}{dt} = 4\pi(10 + r)^2 \frac{dr}{dt}$



Substitute r = 5,

$50 = 4\pi(225)\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{50}{4\pi(225)} = \frac{1}{18\pi} cm / min$

Question140

A water tank has the shape of an inverted right circular cone, whose semi-vertical angle is $\tan^{-1}$. Water is poured into it at a constant rate of 5 cubic meter per minute. Then the rate (in m/min.), at which the level of water is rising at the instant when the depth of water in the tank is 10m; is:
[April 09, 2019 (II)]

Options:

A. $1/15\pi$

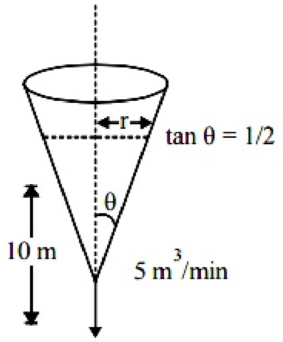
B. $1/10\pi$

C. $2/\pi$

D. $1/5\pi$

Answer: D

Solution:



Given that water is poured into the tank at a constant rate of $5m^3$/ minute.
$\therefore \frac{dv}{dt} = 5m^3 / \text{min}$
Volume of the tank is,
$V = \frac{1}{3}\pi r^2 h$.,......(i)
where r is radius and h is height at any time.
By the diagram,
$\tan\theta = \frac{r}{h} = \frac{1}{2}$
$\Rightarrow h = 2r \Rightarrow \frac{dh}{dt} = \frac{2dr}{dt}$ ....(ii)
Differentiate eq. (i) w.r.t. 't', we get
$\frac{dV}{dt} = \frac{1}{3}\left(\pi 2r\frac{dr}{dt}h + \pi r^2\frac{dh}{dt}\right)$
Putting $h = 10$, $r = 5$ and $\frac{dV}{dt} = 5$ in the above equation.
$5 = \frac{75\pi}{3}\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{5\pi} m / \text{min}.$

Question141

Let $f(x) = e^x - x$ and $g(x) = x^2 - x$, $x \in R$. Then the set of all$x \in R$, where the function $h(x) = (f o g)(x)$ is increasing, is: [April 10, 2019 (I)]

Options:

A. $\left[-1, \frac{-1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

B. $\left[0, \frac{1}{2}\right] \cup [1, \infty)$

C. $[0, \infty)$

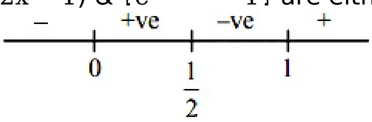
D. $\left[\frac{-1}{2}, 0\right] \cup [1, \infty)$

Answer: B

Solution:

Solution:
Given functions are, $f(x) = e^x - x$ and $g(x) = x^2 - x$
$f(g(x)) = e^{(x^2 - x)} - {}^\wedge 2 - x)$
Given $f(g(x))$ is increasing function.
$\therefore (f(g(x)))' = e^{(x^2 - x)} \times (2x - 1) - 2x + 1$

$(2x - 1)$ & $\left[e^{(x^2 - x)} - 1\right]$ are either both positive or negative



$x \in \left[0, \frac{1}{2}\right] \cup [1, \infty)$

Question142

If the function $f : R - \{1, -1\} \to A$ defined by $f(x) = \frac{x^2}{1 - x^2}$, is surjective, then A is equal to:
[April 09, 2019 (I)]

Options:

A. $R - \{-1\}$

B. [0, ")

C. $R - [-1, 0)$

D. $R - (-1, 0)$

Answer: C

Solution:

Solution:

$f(x) = \frac{x^2}{1 - x^2}$

$\Rightarrow f(-x) = \frac{x^2}{1 - x^2} = f(x)$

$f'(-x) = \frac{2x}{(1 - x^2)^2}$

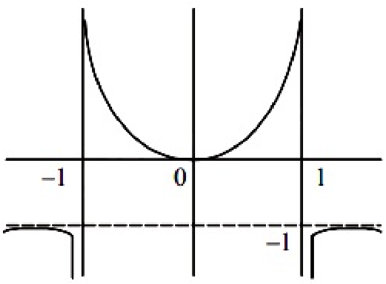
$f(x)$ increases in $x \in (10, \infty)$

Also $f(0) = 0$ and

$\lim_{x \to \pm\infty} f(x) = -1$ and $f(x)$ is even function

Set $A = R - [-1, 0)$

And the graph of function $f(x)$ is



Question143

Let $f : [0 : 2] \to R$ be a twice differentiable function such that $f''(x) > 0$, for all $x \in (0, 2)$. If $\phi(x) = f(x) + f(2 - x)$, then ϕ is :

Options:

A. increasing on (0, 1) and decreasing on (1, 2).

B. decreasing on (0, 2)

C. decreasing on (0, 1) and increasing on (1, 2).

D. increasing on (0, 2)

Answer: C

Solution:

Solution:
$f(x) = f(x) + f(2 - x)$
Now, differentiate w.r.t. x
$f'(x) = f'(x) - f'(2 - x)$
For $f(x)$ to be increasing $f'(x) > 0$
$\Rightarrow f'(x) - f'(2 - x) > 0$
$\Rightarrow f'(x) > f'(2 - x)$
But $f''(x) > 0 \Rightarrow f'(x)$ is an increasing function
Then, $f'(x) > f'(2 - x) > 0$
$\Rightarrow x > 2 - x$
$\Rightarrow x > 1$
Hence, $f(x)$ is increasing on (1,2) and decreasing on (0,1)

Question144

If the tangent to the curve $y = xx^2 - 3$, $x \in R$, $(x \neq \pm\sqrt{3})$, at a point $(\alpha, \beta)(0, 0)$ on it is parallel to the line $2x + 6y - 11 = 0$, then : [April 10, 2019 (II)]

Options:

A. $|6\alpha + 2\beta| = 19$

B. $|6\alpha + 2\beta| = 9$

C. $|2\alpha + 6\beta| = 19$

D. $|2\alpha + 6\beta| = 11$

Answer: A

Solution:

Solution:
Given curve is, $y = \frac{x}{x^2 - 3}$
$\Rightarrow \frac{dy}{dx} = \frac{(x^2 - 3) - x(2x)}{(x^2 - 3)^2} = \frac{-x^2 - 3}{(x^2 - 3)^2}$
$\left.\frac{dy}{dx}\right|_{(\alpha, \beta)} = \frac{\alpha^2 - 3}{(\alpha^2 - 3)^2} = -\frac{2}{6} = -\frac{1}{3}$
$3(\alpha^2 + 3) = (\alpha^2 - 3)^2 \Rightarrow \alpha^2 = 9$
And, $\beta = \frac{x}{a^2 - 3} \Rightarrow \alpha^2 - 3 = \frac{\alpha}{\beta} \Rightarrow \frac{\alpha}{\beta} = 6$

These values of α and β satisfies $|6\alpha + 2\beta| = 19$

Question145

If the tangent to the curve, $y = x^3 + ax - b$ at the point (1,-5) is perpendicular to the line, $-x + y + 4 = 0$, then which one of the following points lies on the curve?
[April 09, 2019 (I)]

Options:

A. (-2,1)

B. (-2,2)

C. (2,-1)

D. (2,-2)

Answer: D

Solution:

Solution:
$y = x^3 + ax - b$
Since, the point (1,-5) lies on the curve.
$\Rightarrow 1 + a - b = -5$
$\Rightarrow a - b = -6$ ......(i)
$\frac{dy}{dx} = 3x^2 + a$
$\left(\frac{dy}{dx}\right)_{\text{at } x=1} = 3 + a$
Since, required line is perpendicular to $y = x - 4$, then slope of tangent at the point $P(1, -5) = -1$
$3 + a = -1$
$a = -4$
$b = 2$
the equation of the curve is $y = x^3 - 4x - 2$ (2,-2) lies on the curve

----

Question146

Let S be the set of all values of x for which the tangent to the curve $y = f(x) = x^3 - x^2 - 2x$ at (x, y) is parallel to the line segment joining the points $(1, f(1))$ and $(-1, f(-1))$, then S is equal to:
[April 09, 2019 (I)]

Options:

A. $\left\{\frac{1}{3}, 1\right\}$

B. $\left\{-\frac{1}{3}, -1\right\}$

D. $\left\{-\frac{1}{3}, 1\right\}$

Answer: D

Solution:

Solution:

$y = f(x) = x^3 - x^2 - 2x$

$\frac{dy}{dx} - 3x^2 - 2x - 2$

$f(1) = 1 - 1 - 2 = -2,\ f(-1) = -1 - 1 + 2 = 0$

Since the tangent to the curve is parallel to the line segment joining the points (1,-2)(-1,0)

Since their slopes are equal

$\Rightarrow 3x^2 - 2x - 2 = \frac{-2 - 0}{2} \Rightarrow x = 1, \frac{-1}{3}$

Hence, the required set S = $\left\{\frac{-1}{3}, 1\right\}$

Question147

The tangent and the normal lines at the point $(\sqrt{3}, 1)$ to the circle $x^2 + y^2 = 4$ and the x -axis form a triangle. The area of this triangle (in square units) is:
[April 08,2019 (II)]

Options:

A. $\frac{4}{\sqrt{3}}$

B. $\frac{1}{3}$

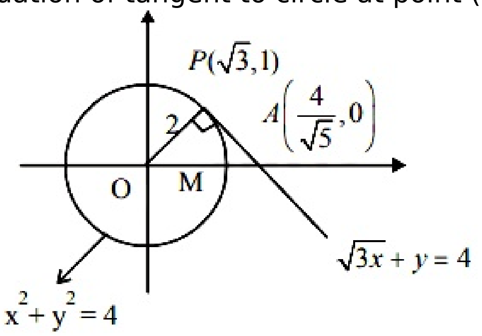
C. $\frac{2}{\sqrt{3}}$

D. $\frac{1}{\sqrt{3}}$

Answer: C

Solution:

Equation of tangent to circle at point $(\sqrt{3}, 1)$ is $\sqrt{3}x + y = 4$



coordinates of the point A = $\left(\frac{4}{\sqrt{3}}, 0\right)$

Area $= \frac{1}{2} \times OA \times PM = \frac{1}{2} \times \frac{4}{\sqrt{3}} \times 1 = \frac{2}{\sqrt{3}}$ sq. units

Question148

If m is the minimum value of k for which the function $f(x) = x\sqrt{kx - x^2}$ is increasing in the interval [0,3] and M is the maximum value of f in [0,3] when k = m, then the ordered pair (m, M) is equal to :
[April 12, 2019 (I)]

Options:

A. $(4, 3\sqrt{2})$

B. $(4, 3\sqrt{3})$

C. $(3, 3\sqrt{3})$

D. $(5, 3\sqrt{6})$

Answer: B

Solution:

Solution:
Given function $f(x) = x\sqrt{kx - x^2} = \sqrt{kx^3 - x^4}$
Differentiating w. r. t. x,
$f'(x) = \frac{(3kx^2 - 4x^3)}{2\sqrt{kx^3 - x^4}} \geq 0$ for $x \in [0, 3]$ [$\because f(x)$ is increasing in [0,3]]
$\Rightarrow 3k - 4x \geq 0 \Rightarrow 3k \geq 4x$
i.e., $3k \geq 4x$ for $x \in [0, 3]$
$\therefore k \geq 4$ i.e., $m = 4$
Putting $k = 4$ in the function, $f(x) = x\sqrt{4x - x^2}$
For max. value, $f'(x) = 0$
i.e. $\frac{12x^2 - 4x^3}{2\sqrt{4x^3 - x^4}} = 0 \Rightarrow x = 3$
$y = 3\sqrt{3}$ i.e., $M = 3\sqrt{3}$

Question149

Let $a_1, a_2, a_3, ...$ be an A. P. with $a_6 = 2$. Then the common difference of this A.P., which maximises the product $a_1 a_4 a_5$, is
[April 10, 2019 (II)]

Options:

A. $\frac{3}{2}$

B. $\frac{8}{5}$

C. $\frac{6}{5}$

Answer: B

Solution:

Solution:

$a_6 = a + 5d = 2$

Here, a is first term of A.P and d is common difference

Let $A = a_1 a_4 a_5 = a(a + 3d)(a + 4d)$

$= a(2 - 2d)(2 - d)$

$A = (2 - 5d)(4 - 6d + 2d^2)$

By $\frac{dA}{dd} = 0$

$(2 - 5d)(-6 + 4d) + (4 - 6d + 2d^2)(-5) = 0$

$-15d^2 + 34d - 16 = 0 \Rightarrow d = \frac{8}{5}, \frac{2}{3}$

For $d = \frac{8}{5}, \frac{d^2A}{dd^2} < 0$

Hence $d = \frac{8}{5}$

Question150

If S_1 and S_2 are respectively the sets of local minimum and local maximum points of the function, $f(x) = 9x^4 + 12x^3 - 36x^2 + 25$, $x \in R$, then

[April 08, 2019 (I)]

Options:

A. $S_1 = \{-2\}; S_2 = \{0, 1\}$

B. $S_1 = \{-2, 0\}; S_2 = \{1\}$

C. $S_1 = \{-2, 1\}; S_2 = \{0\}$

D. $S_1 = \{-1\}; S_2 = \{0, 2\}$

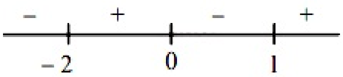
Answer: C

Solution:

Solution:

$f(x) = 9x^4 + 12x^3 - 36x^2 + 25$

$f'(x) = 36[x^3 + x^2 - 2x] = 36x(x - 1)(x + 2)$



Here at -2 &1, $f'(x)$ changes from negative value to positive value.

$\Rightarrow -2$ & 1 are local minimum points. At 0, $f'(x)$ changes from positive value to negative value.

$\Rightarrow 0$ is the local maximum point.

Hence, $S_1 = \{-2, 1\}$ and $S_2 = \{0\}$

Question151

The height of a right circular cylinder of maximum volume inscribed in a sphere of radius 3 is:
[April 08, 2019 (II)]

Options:

A. $\sqrt{6}$

B. $\frac{2}{3}\sqrt{3}$

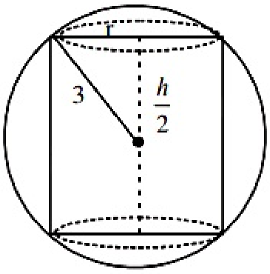
C. $2\sqrt{3}$

D. $\sqrt{3}$

Answer: C

Solution:

Solution:
Let radius of base and height of cylinder be r and h respectively



$\therefore r^2 + \frac{h^2}{4} = 9$ ......(i)
Now, volume of cylinder, $V = \pi r^2 h$
Substitute the value of r^2 from equation (i),
$V = \pi h\left(9 - \frac{h^2}{4}\right) \Rightarrow V = 9\pi h - \frac{\pi}{4}h^3$
Differentiating w.r.t. h
$\frac{dV}{dh} = 9\pi - \frac{3}{4}\pi h^2$
For maxima/minima,
$\frac{dV}{dh} = 0 \Rightarrow h = \sqrt{12}$
and $\frac{d^2V}{dh^2} = -\frac{3}{2}\pi h$
$\left(\frac{d^2V}{dh^2}\right)_{h=\sqrt{12}} < 0$
Volume is maximum when $h = 2\sqrt{3}$

Question152

A 2 m ladder leans against a vertical wall. If the top of the ladder begins to slide down the wall at the rate 25 cm/sec., then the rate (in cm/sec.) at which the bottom of the ladder slides away from the wall on the horizontal ground when the top of the ladder is 1 m above the ground is:
[April 12, 2019 (I)]

A. $25\sqrt{3}$

B. $\frac{25}{\sqrt{3}}$

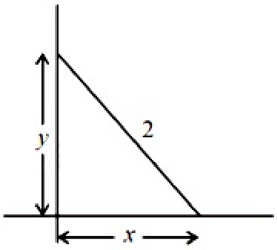
C. $\frac{25}{3}$

D. 25

Answer: B

Solution:

Solution:
According to the question,
$\frac{dy}{dt} = -25$ at $y = 1$
By Pythagoras theorem, $x^2 + y^2 = 4$ ....(i)
When $y = 1 \Rightarrow x = \sqrt{3}$
Diff. equation (i) w. r. t. t,
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$



$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \Rightarrow \sqrt{3}\frac{dx}{dt} + (-25) = 0$
$\Rightarrow \frac{dx}{dt} = \frac{25}{\sqrt{3}}$ cm / s

Question153

If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is :
[2018]

Options:

A. $\frac{7}{2}$

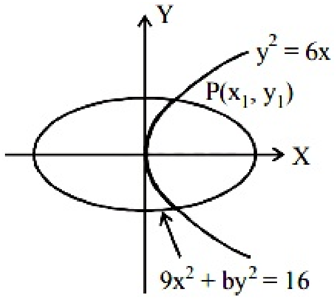
B. 4

C. $\frac{9}{2}$

D. 6

Answer: C

Solution:



Since, point of intersection is on both the curves, then
$y_1^{\ 2} = 6x_1$ ......(i)
and $9x_1^{\ 2} + by_1^{\ 2} = 16$ .....(ii)
Now, find the slope of tangent to both the curves at the point of intersection $P(x_1, y_1)$
For slope of curves:
Curve (i):
$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = m_1 = \frac{3}{y_1}$
Curve (ii):
and $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = m_2 = -\frac{9x_1}{by_1}$
Since, both the curves intersect each other at right angle then,
$m_1 m_2 = -1 \Rightarrow \frac{27x_1}{by_1^{\ 2}} = 1 \Rightarrow b = 27\frac{x_1}{y_1^{\ 2}}$
$\therefore$ from equation (i), $b = 27 \times \frac{1}{6} = \frac{9}{2}$

Question154

Let P be a point on the parabola $x^2 = 4y$. If the distance of P from the centre of the circle, $x^2 + y^2 + 6x + 8 = 0$ is minimum, then the equation of the tangent to the parabola at P, is [Online April 16, 2018]

Options:

A. $x + 4y - 2 = 0$

B. $x + 2y = 0$

C. $x + y + 1 = 0$

D. $x - y + 3 = 0$

Answer: C

Solution:

Solution:
Let $P(2t, t^2)$ be any point on the parabola.
Centre of the given circle $C = (-g, -f) = (-3, 0)$
For PC to be minimum, it must be the normal to the parabola at P.
Slope of line PC $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{t^2 - 0}{2t + 3}$
Also, slope of tangent to parabola at $P = \frac{dy}{dx} = \frac{x}{2} = t$
$\therefore$ Slope of normal $= \frac{-1}{t}$

$\Rightarrow t^3 + 2t + 3 = 0$
$\Rightarrow (t+1)(t^2 - t + 3) = 0$
$\therefore$ Real roots of above equation is $t = -1$
Coordinate of $P = (2t, t^2) = (-2, 1)$
Slope of tangent to parabola at $P = t = -1$
Therefore, equation of tangent is:
$(y - 1) = (-1)(x + 2)$
$\Rightarrow x + y + 1 = 0$

Question155

If the tangents drawn to the hyperbola $4y^2 = x^2 + 1$ intersect the co-ordinate axes at the distinct points A and B, then the locus of the mid point of AB is [Online April 15, 2018]

Options:

A. $x^2 - 4y^2 + 16x^2y^2 = 0$

B. $4x^2 - y^2 + 16x^2y^2 = 0$

C. $4x^2 - y^2 - 16x^2y^2 = 0$

D. $x^2 - 4y^2 - 16x^2y^2 = 0$

Answer: D

Solution:

Solution:
Equation of hyperbola is:

$4y^2 = x^2 + 1$
$\Rightarrow -x^2 + 4y^2 = 1$
$\Rightarrow -\frac{x^2}{1^2} + \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1$
$\therefore a = 1, b = \frac{1}{2}$
Now, tangent to the curve at point (x_1, y_1) is given by
$4 \times 2y_1 \frac{dy}{dx} = 2x_1$
$\Rightarrow \frac{dy}{dx} = \frac{2x_1}{8y_1} = \frac{x_1}{4y_1}$
Equation of tangent at (x_1, y_1) is
$y = mx + c$
$\Rightarrow y = \frac{x_1}{4y_1} . x + c$
As tangent passes through (x_1, y_1)
$\therefore y_1 = \frac{x_1 x_1}{4y_1} + c$
$\Rightarrow C = \frac{4{y_1}^2 - {x_1}^2}{4y_1} = \frac{1}{4y_1}$
Therefore, $y = \frac{x_1}{4y_1}x + \frac{1}{4y_1} \Rightarrow 4y_1 y = x_1 x + 1$
which intersects x axis at $A\left(\frac{-1}{\ldots}, 0\right)$ and y axis at

$B\left(0, \frac{1}{4y_1}\right)$

Let midpoint of AB is (h, k)

$\therefore h = \frac{-1}{2x_1}$

$\Rightarrow x_1 = \frac{-1}{2h}$ & $y_1 = \frac{1}{8k}$

Thus, $4\left(\frac{1}{8k}\right)^2 = \left(\frac{-1}{2h}\right)^2 + 1$

$\Rightarrow \frac{1}{16k^2} = \frac{1}{4h^2} + 1$

$\Rightarrow 1 = \frac{16k^2}{4h^2} + 16k^2$

$\Rightarrow h^2 = 4k^2 + 16h^2k$

So, required equation is

$x^2 - 4y^2 - 16x^2y^2 = 0$

Question156

If β is one of the angles between the normals to the ellipse, $x^2 + 3y^2 = 9$ at the points $(3\cos\theta, \sqrt{3}\sin\theta)$ and $(-3\sin\theta, \sqrt{3}\cos\theta)$; $\in\left(0, \frac{\pi}{2}\right)$; then $\frac{2\cot\beta}{\sin 2\theta}$ is equal to
[Online April 15, 2018]

Options:

A. $\sqrt{2}$

B. $\frac{2}{\sqrt{3}}$

C. $\frac{1}{\sqrt{3}}$

D. $\frac{\sqrt{3}}{4}$

Answer: B

Solution:

Solution:

Since, $x^2 + 3y^2 = 9$

$\Rightarrow 2x + 6y\, dy/dx = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{3y}$

Slope of normal is $-\frac{dx}{dy} = \frac{3y}{x}$

$\Rightarrow \left(-\frac{dx}{dy}\right)_{(3\cos\theta, \sqrt{3}\sin\theta)} = \frac{3\sqrt{3}\sin\theta}{3\cos\theta} = \sqrt{3}\tan\theta = m_1$

& $\left(-\frac{dx}{dy}\right)_{(-3\sin\theta, \sqrt{3}\cos\theta)}$

$= \frac{3\sqrt{3}\cos\theta}{-3\sin\theta} = -\sqrt{3}\cot\theta = m_2$

As, β is the anagle between the normals to the given ellipse then

$\tan\beta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

So, $\tan\beta = \frac{\sqrt{3}}{2}\left|\tan\theta + \cot\theta\right|$

$\Rightarrow \frac{1}{\cot\beta} = \frac{\sqrt{3}}{2}\left|\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right|$

$\Rightarrow \frac{1}{\cot\beta} = \frac{\sqrt{3}}{2}\left|\frac{1}{\sin\theta\cos\theta}\right|$

$\Rightarrow \frac{1}{\cot\beta} = \frac{\sqrt{3}}{\sin 2\theta} \Rightarrow \frac{2\cot\beta}{\sin 2\theta} = \frac{2}{\sqrt{3}}$

Question157

A normal to the hyperbola, $4x^2 - 9y^2 = 36$ meets the coordinate axes x and y at A and B, respectively. If the parallelogram OABP(O being the origin) is formed, then the locus of P is
[Online April 15, 2018]

Options:

A. $4x^2 - 9y^2 = 121$

B. $4x^2 + 9y^2 = 121$

C. $9x^2 - 4y^2 = 169$

D. $9x^2 + 4y^2 = 169$

Answer: C

Solution:

Solution:
Given, $4x^2 - 9y^2 = 36$
After differentiating w.r.t. x, we get
$4.2 . x - 9.2 . y . \frac{dy}{dx} = 0$

$\Rightarrow$ Slope of tangent $= \frac{dy}{dx} = \frac{4x}{9y}$

So, slope of normal $= \frac{-9y}{4x}$

Now, equation of normal at point (x_0, y_0) is given by$y - y_0 = \frac{-9y_0}{4x_0}(x - x_0)$

As normal intersects X axis at A, Then

$A \equiv \left(\frac{13x_0}{9}, 0\right)$ and $B \equiv \left(0, \frac{13y_0}{4}\right)$

As OABP is a parallelogram

midpoint of OB $\equiv \left(0, \frac{13y_0}{8}\right) \equiv$ Midpoint of AP

So, $P(x, y) \equiv \left(\frac{-13x_0}{9}, \frac{13y_0}{4}\right)$

$\because (x_0, y_0)$ lies on hyperbola, therefore

$4(x_0)^2 - 9(y_0)^2 = 36$ ....(ii)

From equation (i): $x_0 = \frac{-9x}{13}$ and $y_0 = \frac{4y}{13}$

From equation (ii), we get

$9x^2 - 4y^2 = 169$

Hence, locus of point P is :$9x^2 - 4y^2 = 169$

Question158

Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}, x \in R - \{-1, 0, 1\}$
If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of h(x) is:
[2018]

Options:

A. -3

B. $-2\sqrt{2}$

C. $2\sqrt{2}$

D. 3

Answer: C

Solution:

Solution:

Here, $h(x) = \frac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} = \left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}}$

When $x - \frac{1}{x} < 0$

$\therefore x - \frac{1}{x} + \frac{2}{x - \frac{1}{x}} \leq -2\sqrt{2}$

Hence, $-2\sqrt{2}$ will be local maximum value of $h(x)$

When $x - \frac{\sqrt{1}}{x} > 0$

$\therefore x - \frac{1}{x} + \frac{2}{x - \frac{1}{x}} \geq 2\sqrt{2}$

Hence, $2\sqrt{2}$ will be local minimum value of $h(x)$.

Question159

Let M and m be respectively the absolute maximum and the absolute minimum values of the function, $f(x) = 2x^3 - 9x^2 + 12x + 5$ in the interval [0,3] . Then M − m is equal to
[Online April 16, 2018]

Options:

A. 1

B. 5

C. 4

D. 9

Solution:

Here, $f(x) = 2x^3 - 9x^2 + 12x + 5$
$\Rightarrow f'(x) = 6x^2 - 18x + 12 = 0$
For maxima or minima put $f'(x) = 0$
$\Rightarrow x^2 - 3x + 2 = 0$
$\Rightarrow x = 1$ or $x = 2$
Now, $f''(x) = 12x - 18$
$\Rightarrow f''(1) = 12(1) - 18 = -6 < 0$
Hence, $f(x)$ has maxima at $x = 1$
$\therefore$ maximum value $= M = f(1) = 2 - 9 + 12 + 5 = 10.$
And, $f''(2) = 12(2) - 18 = 6 > 0.$
Hence, $f(x)$ has minima at $x = 2.$
$\therefore$ minimum value $= m = f(2)$
$= 2(8) - 9(4) + 12(2) + 5 = 9$
$\therefore M - m = 10 - 9 = 1$

Question160

If a right circular cone having maximum volume, is inscribed in a sphere of radius 3cm, then the curved surface area (in cm^2) of this cone is
[Online April 15, 2018]

Options:

A. $8\sqrt{3}\pi$

B. $6\sqrt{2}\pi$

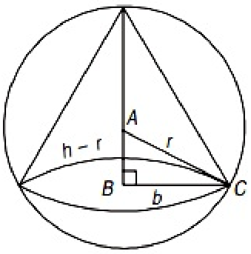
C. $6\sqrt{3}\pi$

D. $8\sqrt{2}\pi$

Answer: A

Solution:

Solution:
Sphere of radius $r = 3cm$ Let b, h be base radius and height of cone respectively.
So, volume of cone $= \frac{1}{2}\pi b^2 h$



In right angled ΔABC by Pythagoras theorem
$(h - r)^2 + b^2 = r^2$ ......(i)
$\Rightarrow b^2 = r^2 - (h - r)^2 = r^2 - (h^2 - 2hr + r^2) = 2hr - h^2$
$\therefore$ Volume $(v) = \frac{1}{3}\pi h[2hr - h^2] = \frac{1}{3}[2h^2 r - h^3]$
$\frac{dv}{dh} = \frac{1}{3}[4hr - 3h^2] = 0 \Rightarrow h(4r - 3h) = 0$

$\frac{d^2v}{dh^2} = \frac{1}{3}[4r - 6h]$

At $h = \frac{4r}{3}, \frac{d^2v}{dh^2} = \frac{1}{3}\left[4r - \frac{4r}{3} \times 6\right] = \frac{1}{3}[4r - 8r] < 0$

⇒ maximum volume ocurs at $h = \frac{4r}{3} = \frac{4}{3} \times 3 = 4cm$

As from (i),

$(h - r)^2 + b^2 = r^2$

$\Rightarrow b^2 = 2hr - h^2 = 2 . \frac{4r}{3}r - \frac{16r^2}{9} = \frac{8r^2}{3} - \frac{16r^2}{9}$

$= \frac{(24 - 16)r^2}{9} = \frac{8r^2}{9}$

$\Rightarrow b = \frac{2\sqrt{2}}{3}r = 2\sqrt{2}cm$

Therefore curved surface area $= \pi bl$

$= \pi b\sqrt{h^2 + r^2} = \pi 2\sqrt{2}\sqrt{4^2 + 8} = 8\sqrt{3}\pi cm^2$

Question161

The function f defined by $f(x) = x^3 - 3x^2 + 5x + 7$, is: [Online April 9, 2017]

Options:

A. increasing in R.

B. decreasing in R.

C. decreasing in (0, ∞) and increasing in (– ∞ , 0).

D. increasing in (0, ∞) and decreasing in (– ∞, 0).

Answer: A

Solution:

Solution:

$f(x) = x^3 - 3x^2 + 5x + 7$

For increasing

$f'(x) = 3x^2 - 6x + 5 > 0$

$\Rightarrow x \in R$

For decreasing

$f'(x) = 3x^2 - 6x + 5 < 0$

Question162

The normal to the curve $y(x - 2)(x - 3) = x + 6$ at the point where the curve intersects the y-axis passes through the point: [2017]

Options:

A. $\left(\frac{1}{\ } \frac{1}{\ }\right)$

B. $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

C. $\left(\frac{1}{2}, \frac{1}{2}\right)$

D. $\left(\frac{1}{2}, -\frac{1}{3}\right)$

Answer: C

Solution:

Solution:

We have $y = \frac{x+6}{(x-2)(x-3)}$

At y -axis, $x = 0 \Rightarrow y = 1$

On differentiating, we get

$\frac{dy}{dx} = \frac{(x^2-5x+6)(1)-(x+6)(2x-5)}{(x^2-5x+6)^2}$

$\frac{dy}{dx} = 1$ at point (0,1)

$\therefore$ Slope of normal $= -1$

Now equation of normal is $y - 1 = -1(x - 0)$

$\Rightarrow y - 1 = -x$

$x + y = 1$

$\therefore \left(\frac{1}{2}, \frac{1}{2}\right)$ satisfy it.

Question163

The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directices is $x = -4$, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is :
[2017]

Options:

A. $x + 2y = 4$

B. $2y - x = 2$

C. $4x - 2y = 1$

D. $4x + 2y = 7$

Answer: C

Solution:

Solution:

Eccentricity of ellipse $= \frac{1}{2}$

Now, $-\frac{a}{e} = -4 \Rightarrow a = 4 \times \frac{1}{2} = 2 \Rightarrow a = 2$

$= 4 \times \frac{3}{4} = 3$

∴ Equation of ellipse is

$\frac{x^2}{4} + \frac{y^2}{3} = 1$

Now differentiating, we get

$\Rightarrow \frac{x}{2} + \frac{2y}{3} \times y' = 0 \Rightarrow y' = -\frac{3x}{4y}$

$y'|_{(1, 3/2)}| = -\frac{3}{4} \times \frac{2}{3} = -\frac{1}{2}$

Slope of normal $= 2$

∴ Equation of normal at $\left(1, \frac{3}{2}\right)$ is

$y - \frac{3}{2} = 2(x - 1) \Rightarrow 2y - 3 = 4x - 4$

$\therefore 4x - 2y = 1$

Question164

A tangent to the curve, y = f (x) at P(x, y) meets x -axis at A and y -axis at B. If AP : BP = 1 : 3 and f (a) = 1, then the curve also passes through the point:
[Online April 9, 2017]

Options:

A. $\left(\frac{1}{3}, 24\right)$

B. $\left(\frac{1}{2}, 4\right)$

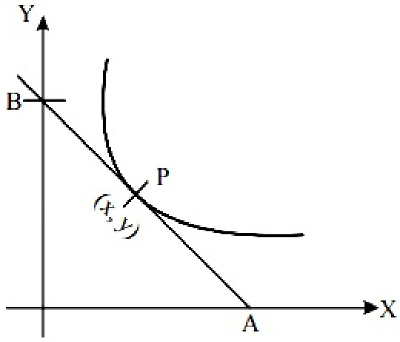
C. $\left(2, \frac{1}{8}\right)$

D. $\left(3, \frac{1}{28}\right)$

Answer: C

Solution:

Solution:



Let $y = f(x)$ be a curve

slope of tangent $= f'(x)$

Equation of tangent $(Y - y) = f'(x)(X - x)$

Put $Y = 0$

$\Rightarrow X = \left(x - \frac{y}{\ }\right)$

$\Rightarrow Y = y - xf'(x)$
$\Rightarrow A = \left(x - \frac{y}{f'(x)}, 0\right)$
and $B = (0, y - xf'(x))$
$\therefore AP : PB = 1 : 3$
$\Rightarrow x = \frac{3}{4}\left(x - \frac{y}{f'(x)}\right)$
$\Rightarrow x = \frac{-3y}{f'(x)} \Rightarrow \frac{dy}{dx} = \frac{-3y}{x}$
$\frac{dy}{y} = \frac{-3dx}{x} \Rightarrow y = \frac{C}{x^3}$
$\because f(a) = 1 \Rightarrow C = 1$
$\therefore y = \frac{1}{x^3}$ is required curve and $\left(2, \frac{1}{8}\right)$ passingthrough $y = \frac{1}{x^3}$

Question165

The tangent at the point (2,-2) to the curve, $x^2y^2 - 2x = 4(1 - y)$ does not pass through the point:
[Online April 8, 2017]

Options:

A. $\left(4, \frac{1}{3}\right)$

B. (8,5)

C. (-4,-9)

D. (-2,-7)

Answer: D

Solution:

Solution:
$x^2y^2 - 2x = 4 - 4y$
Differentiate w.r.t. 'x'
$2xy^2 + 2y \cdot x^2 \cdot \frac{dy}{dx} - 2 = -4 \cdot \frac{dy}{dx}$
$\Rightarrow \frac{dy}{dx}(2y \cdot x^2 + 4) = 2 - 2x \cdot y^2$
$\Rightarrow \frac{dy}{dx}\Big|_{2,-2} = \frac{2 - 2 \times 2 \times 4}{2(-2) \times 4 + 4} = \frac{-14}{-12} = \frac{7}{6}$
$\therefore$ Equation of tangent is $(y + 2) = \frac{7}{6}(x - 2)$ or $7x - 6y = 26$
$\therefore (-2, -7)$ does not passes through the required tangent.

Question166

Twenty metres of wire is available for fencing off a flowerbed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed, is:
[2017]

Options:

A. 30

B. 12.5

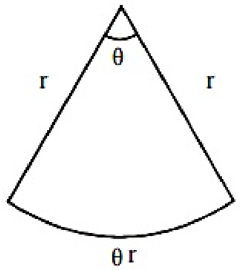
C. 10

D. 25

Answer: D

Solution:

Solution:



We have

Total length $= r + r + r\theta = 20$

$\Rightarrow 2r + r\theta = 20$

$\Rightarrow \theta = \frac{20 - 2r}{r}$ ....(i)

$A =$ Area $= \frac{\theta}{2\theta} \times \theta r^2$

$= \frac{1}{2} r^2 \theta = \frac{1}{2} r^2 \left(\frac{20 - 2r}{r} \right)$

$A = 10r - r^2$

For A to be maximum

$\frac{dA}{dr} = 0 \Rightarrow 10 - 2r = 0$

$\Rightarrow r = 5$

$\frac{d^2A}{dr^2} = -2 < 0$

$\therefore$ For $r = 5$ A is maximum

From (i)

$\theta = \frac{20 - 2(5)}{5} = \frac{10}{5} = 2$

$A = \frac{2}{2\pi} \times \pi (5)^2 = 25 \, \text{sq} . \text{m}$

Question167

Let $f(x) = \sin^4 x + \cos^4 x$. Then f is an increasing function in the interval : [JEE Mains 2016]

Options:

A. $\left] \frac{5\pi}{8}, \frac{3\pi}{4} \right]$

B. $\left] \frac{\pi}{2}, \frac{5\pi}{8} \right]$

D. $\left]0, \frac{\pi}{4}\right]$

Answer: C

Solution:

Solution:

$f(x) = \sin^4 x + \cos^4 x$

$f'(x) = 4\sin^3 x \cos x + 4\cos^3 x(-\sin x)$

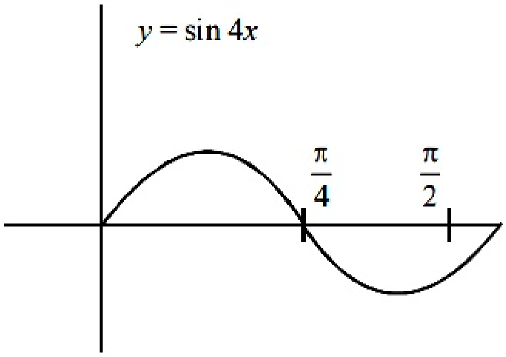
$= 4 \sin x \cos x(\sin^2 x - \cos^2 x)$

$= -2 \sin 2x \cos 2x = -\sin 4x$

f (x) is increasing when $f'(x) > 0$

$\Rightarrow -\sin 4x > 0 \Rightarrow \sin 4x < 0$

$\Rightarrow x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$



Question168

Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x \in \left(0, \frac{\pi}{2}\right)$

A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point: [2016]

Options:

A. $\left(\frac{\pi}{6}, 0\right)$

B. $\left(\frac{\pi}{4}, 0\right)$

C. (0,0)

D. $\left(0, \frac{2\pi}{3}\right)$

Answer: D

Solution:

Solution:

$f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$

$= \tan^{-1}\left(\sqrt{\frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{\left(\sin\frac{x}{x}-\cos\frac{x}{2}\right)^2}}\right) = \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan x 2}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)$

$\Rightarrow y = \frac{\pi}{4}+\frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}$

Slope of normal $= \frac{-1}{\left(\frac{dy}{dx}\right)} = -2$

Equation of normal at $\left(\frac{\pi}{6}, \frac{\pi}{4}+\frac{\pi}{12}\right)$

$y - \left(\frac{\pi}{4}+\frac{\pi}{12}\right) = -2\left(x-\frac{\pi}{6}\right)$

$y - \frac{4\pi}{12} = -2x + \frac{2\pi}{6}$

$y - \frac{\pi}{3} = -2x + \frac{\pi}{3}$

$y = -2x + \frac{2\pi}{3}$

This equation is satisfied only by the point $\left(0, \frac{2\pi}{3}\right)$

Question169

Let C be a curve given by $y(x) = 1 + \sqrt{4x-3}$, $x > \frac{3}{4}$. If P is a point on C, such that the tangent at P has slope $\frac{2}{3}$, then a point through which the normal at P passes, is :
[Online April 10, 2016]

Options:

A. (1,7)

B. (3,-4)

C. (4,-3)

D. (2,3)

Answer: A

Solution:

Solution:
$\frac{dy}{dx} = \frac{1}{2\sqrt{4x-3}} \times 4 = \frac{2}{3}$
$\Rightarrow 4x - 3 = 9$
$\Rightarrow x = 3$
So, $y = 4$
Equation of normal at $P(3, 4)$ is
$y - 4 = -\frac{3}{2}(x-3)$
i.e. $2y - 8 = -3x + 9$
$\Rightarrow 3x + 2y - 17 = 0$
This line is satisfied by the point (1,7)

Question170

If the tangent at a point P, with parameter t, on the curve $x = 4t^2 + 3$, $y = 8t^3 - 1$, $t \in R$, meets the curve again at a point Q, then the coordinates of Q are :
[Online April 9, 2016]

Options:

A. $(16t^2 + 3, -64t^3 - 1)$

B. $(4t^2 + 3, -8t^3 - 2)$

C. $(t^2 + 3, t^3 - 1)$

D. $(t^2 + 3, -t^3 - 1)$

Answer: D

Solution:

Solution:
$P(4t^2 + 3, 8t^3 - 1)$
$\frac{dy/dt}{dt/dt} = \frac{dy}{dx} = 3t$ (slope of tangent at P)
Let $Q = (4\lambda^2 + 3, 8\lambda^3 - 1)$
slope of PQ = 3t
$\frac{8t^3 - 8\lambda^3}{4t^2 - 4\lambda^2} = 3t$
$\Rightarrow t^3 - 3\lambda^2 t + 2\lambda^3 = 0$
$(t - \lambda) . (t^2 + t\lambda - 2\lambda^2) = 0$
$(t - \lambda)^2 . (t + 2\lambda) = 0$
$t = \lambda$ (or) $\lambda = \frac{-t}{2}$
$\therefore Q[t^2 + 3, -t^3 - 1]$

Question171

A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then:
[2016]

Options:

A. $x = 2r$

B. $2x = r$

C. $2x = (\pi + 4)r$

D. $(4 - \pi)x = \pi r$

Answer: A

Solution:

Solution:

$4x + 2\pi r = 2 \Rightarrow 2x + \pi r = 1$

$S = x^2 + \pi r^2$

$S = \left(\frac{1 - \pi r}{2}\right)^2 + \pi r^2$

$\frac{dS}{dr} = 2\left(\frac{1 - \pi r}{2}\right)\left(\frac{-\pi}{2}\right) + 2\pi r$

$\Rightarrow \frac{-\pi}{2} + \frac{\pi^2 r}{2} + 2\pi r = 0 \Rightarrow r = \frac{1}{\pi + 4}$

$\Rightarrow x = \frac{2}{\pi + 4} \Rightarrow x = 2r$

Question172

The minimum distance of a point on the curve $y = x^2 - 4$ from the origin is:
[Online April 9, 2016]

Options:

A. $\frac{\sqrt{15}}{2}$

B. $\sqrt{\frac{19}{2}}$

C. $\sqrt{\frac{15}{2}}$

D. $\frac{\sqrt{19}}{2}$

Answer: A

Solution:

Solution:

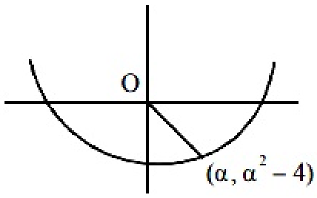
$D = \sqrt{\alpha^2 + (\alpha^2 - 4)^2}$

$D^2 = \alpha^2 + \alpha^4 + 16 - 8\alpha^2 = \alpha^4 - 7\alpha^2 + 16$

$\frac{dD^2}{d\alpha} = 4\alpha^3 - 14\alpha = 0$

$2\alpha(2\alpha^2 - 7) = 0$

$\alpha^2 = \frac{7}{2}$



$D^2 = \frac{49}{4} - \frac{49}{2} + 16 = -\frac{49}{4} + 16 = \frac{15}{4}$

$D = \frac{\sqrt{15}}{2}$

Question173

The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at (1,1) [2015]

Options:

A. meets the curve again in the third quadrant.

B. meets the curve again in the fourth quadrant.

C. does not meet the curve again.

D. meets the curve again in the second quadrant.

Answer: B

Solution:

Solution:
Given curve is
$x^2 + 2xy - 3y^2 = 0$ .....(i)
Differentiate w.r.t.x
$2x + 2x\frac{dy}{dx} + 2y - 6y\frac{dy}{dx} = 0$
$\left(\frac{dy}{dx}\right)_{(1,1)} = 1$
Equation of normal at (1,1) is
$y = 2 - x$ ......(ii)

Solving eqs. (i) and (ii), we get
$x = 1, 3$
Point of intersection (1,1),(3,-1)
Normal cuts the curve again in 4 th quadrant.

Question174

The equation of a normal to the curve, $\sin y = x \sin\left(\frac{\pi}{3} + y\right)$ at $x = 0$, is [Online April 11, 2015]

Options:

A. $2x - \sqrt{3}y = 0$

B. $2x + \sqrt{3}y = 0$

C. $2y - \sqrt{3}x = 0$

D. $2y + \sqrt{3}x = 0$

Answer: B

Solution:

Given curve is $\sin y = x \sin\left(\frac{\pi}{3} + y\right)$
Diff with respect to x, we get

$$\cos y \frac{dy}{dx} = \sin\left(\frac{\pi}{3} + y\right) + x\cos\left(\frac{\pi}{3} + y\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\left(\frac{\pi}{3} + y\right)}{\cos y - x\cos\left(\frac{\pi}{3} + y\right)}$$

$\frac{dy}{dx}$ at $(0, 0) = \frac{\sqrt{3}}{2}$

$\Rightarrow$ Equation of normal is $y - 0 = -\frac{2}{\sqrt{3}}(x - 0)$

$\Rightarrow 2x + \sqrt{3}y = 0$

Question175

If the tangent to the conic, $y - 6 = x^2$ at (2,10) touches the circle, $x^2 + y^2 + 8x - 2y = k$ (for some fixed k) at a point (α, β); then (α, β) is [Online April 10, 2015]

Options:

A. $\left(-\frac{7}{17}, \frac{6}{17}\right)$

B. $\left(-\frac{4}{17}, \frac{1}{17}\right)$

C. $\left(-\frac{6}{17}, \frac{10}{17}\right)$

D. $\left(-\frac{8}{17}, \frac{2}{17}\right)$

Answer: D

Solution:

Solution:
$x^2 - y + 6 = 0$

$2x - \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = 2x$

$\left.\frac{dy}{dx}\right|_{(x, y) = (2, 10)} = 4$

equation of tangent
$y - 10 = 4(x - z)$
$4x - y + z = 0$
tangent passes through (α, β)
$4\alpha - \beta + z = 0 \Rightarrow \beta = 4\alpha + z$ .....(i)
and $2x + 2yy' + 8 - 2y' = 0$

$y' = \frac{2x + 8}{2 - 2y} = \frac{2\alpha + 8}{2 - 2\beta} = 4$ ....(ii)

from (i) and (ii)

$\alpha = \frac{-8}{17}, \beta = \frac{2}{17}$

$\left(-\frac{8}{17}, \frac{2}{17}\right)$

Question176

The distance, from the origin, of the normal to the curve,$x = 2\cos t + 2t\sin t$, $y = 2\sin t - 2t\cos t$ at $t = \frac{\pi}{4}$, is :
[Online April 10, 2015]

Options:

A. 2

B. 4

C. $\sqrt{2}$

D. $2\sqrt{2}$

Answer: A

Solution:

Solution:
Given that
$x = 2\cos t + 2t\sin t$
so, $\frac{dx}{dt} = -2\sin t + 2[t\cos t + \sin t]$
$\frac{dy}{dt} = 2\cos t - 2[-t\sin t + \cos t]$
$\frac{dy}{dx} = 2t\sin t$
$\frac{dy}{dx} = \frac{2t\sin t}{2t\cos t}$
$\frac{dy}{dx} = \tan t$
$\left(\frac{dy}{dx}\right)_{t = \pi/4} = 1$
so the slope of the normal is -1
At $t = \pi/4 x = \sqrt{2} + \frac{\pi}{2\sqrt{2}}$ and $y = \sqrt{2} - \pi/2\sqrt{2}$
the equation of normal is
$[y - (\sqrt{2} - \pi/2\sqrt{2})] = -1[(x - (\sqrt{2} + \pi/2\sqrt{2}))]$
$y - \sqrt{2} + \frac{\pi}{2\sqrt{2}} = -x + \sqrt{2} + \pi/2\sqrt{2}$
$x + y = 2\sqrt{2}$, so the distance from the origin is 2

Question177

Let k and K be the minimum and the maximum values of the function $f(x) = \frac{(1+x)^{0.6}}{1+x^{0.6}}$ in [0,1] respectively, then the ordered pair (k, K) is equal to :
[Online April 11, 2015]

Options:

A. $(2^{-0.4}, 1)$

B. $(2^{-0.6}, 1)$

C. $(2^{-0.4}, 2^{0.6})$

D. $(1, 2^{0.6})$

Answer: A

Solution:

Solution:

Let $f(x) = \frac{(1+x)^{\frac{3}{5}}}{1+x^{\frac{3}{5}}}$ and $x \in [0, 1]$

$$\Rightarrow f'(x) = \frac{\left(1+x^{\frac{3}{5}}\right)\frac{3}{5}(1+x)^{-\frac{2}{5}} - \frac{3}{5}(1+x)^{\frac{3}{5}}\left(x^{\frac{-2}{5}}\right)}{\left(1+x^{\frac{3}{5}}\right)^2}$$

$$= \frac{3}{5}\left[\left(1+x^{\frac{3}{5}}\right)(1+x)^{-\frac{2}{5}} - (1+x)^{\frac{3}{5}}x^{\frac{-2}{5}}\right]$$

$$= \frac{3}{5}\left[\frac{1+x^{\frac{3}{5}}}{(1+x)^{\frac{2}{5}}} - \frac{(1+x)^{\frac{3}{5}}}{x^{\frac{2}{5}}}\right]$$

$$= \frac{x^{\frac{2}{5}} + x - 1 - x}{x^{\frac{2}{5}}(1+x)^{\frac{2}{5}}} = \frac{x^{\frac{2}{5}} - 1}{x^{\frac{2}{5}}(1+x)^{\frac{2}{5}}} < 0$$

Also, $f(0) = 1 \Rightarrow f(x) \in [2^{-0.4}, 1]$
$f(a) = 2^{-0.4}$

Question178

From the top of a 64 metres high tower, a stone is thrown upwards vertically with the velocity of 48m / s. The greatest height (in metres) attained by the stone, assuming the value of the gravitational acceleration $g = 32ms^2$, is:
[Online April 11, 2015]

Options:

A. 128

B. 88

C. 112

D. 100

Answer: D

Solution:

Solution:
Let 'u' be the velocity

Now, we know $v^2 = u^2 - 2gh$
$\Rightarrow 0 = (48)^2 - 2(32)h \Rightarrow h = 36$
Maximum height $= 36 + 64 = 100$mt

Question179

If the volume of a spherical ball is increasing at the rate of 4π cc/sec, then the rate of increase of its radius (in cm/sec), when the volume is 288π cc,
[Online April 19, 2014]

Options:

A. $\frac{1}{6}$

B. $\frac{1}{9}$

C. $\frac{1}{36}$

D. $\frac{1}{24}$

Answer: C

Solution:

Solution:
Volume of sphere $V = \frac{4}{3}\pi r^3$ ......(i)
$\frac{dv}{dt} = \frac{4}{3} . 3\pi r^2 . \frac{dr}{dt}$
$4\pi = 4\pi r^2 . \frac{dr}{dt}$
$\frac{1}{r^2} = \frac{dr}{dt}$
Since, $V = 288\pi$, therefore from (i), we have
$288\pi = \frac{4}{3}\pi(r^3) \Rightarrow \frac{288 \times 3}{4} = r^3$
$\Rightarrow 216 = r^3$
$\Rightarrow r = 6$
Hence, $\frac{dr}{dt} = \frac{1}{36}$.

Question180

Two ships A and B are sailing straight away from a fixed point O along routes such that ∠AOB is always 120° . At a certain instance, OA = 8 km, OB = 6 km and the ship A is sailing at the rate of 20 km/hr while the ship B sailing at the rate of 30 km/hr. Then the distance between A and B is changing at the rate (in km/hr):
[Online April 11, 2014]

A. $\frac{260}{\sqrt{37}}$

B. $\frac{260}{37}$

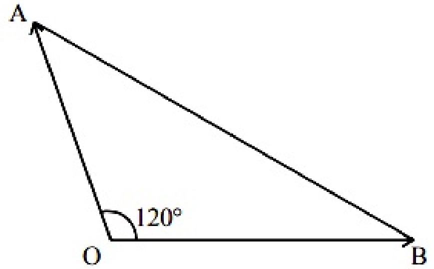
C. $\frac{80}{\sqrt{37}}$

D. $\frac{80}{37}$

Answer: A

Solution:

Solution:



Let OA = xkm, OB = ykm, AB = R

$(AB)^2 = (OA)^2 + (OB)^2 - 2(OA)(OB)\cos 120°$

$R^2 = x^2 + y^2 - 2xy\left(-\frac{1}{2}\right) = x^2 + y^2 + xy$ ......(i)

R at x = 6km, and y = 8km

$R = \sqrt{6^2 + 8^2 + 6 \times 8} = 2\sqrt{37}$

Differentiating equation (i) with respect to t

$2R\frac{dR}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + \left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$

$= \frac{1}{2R}[2 \times 8 \times 20 + 2 \times 6 \times 30 + (8 \times 30 + 6 \times 20)]$

$\frac{dR}{dt} = \frac{1}{2 \times 2\sqrt{37}}[1040] = \frac{260}{\sqrt{37}}$

Question181

Let f and g be two differentiable functions on R such that f'(x) > 0 and g'(x) < 0 for all x ∈ R . Then for all x:
[Online April 12, 2014]

Options:

A. f (g (x)) > f (g (x – 1))

B. f (g (x)) > f (g (x + 1))

C. g(f (x)) > g (f (x – 1))

D. g(f (x)) < g (f (x + 1))

Answer: B

Solution:

Solution:
Since $f'(x) > 0$ and $g'(x) < 0$, therefore $f(x)$ is increasing function and g(x) is decreasing function.
$\Rightarrow f(x+1) > f(x)$ and $g(x+1) $\⇒g[f(x+1)] Hence option (b) is correct.

Question182

For the curve $y = 3\sin\theta\cos\theta$, $x = e^{\theta}\sin\theta$, $0 \le \theta \le \pi$, the tangent is parallel to x -axis when θ is:
[Online April 11, 2014]

Options:

A. $\frac{3\pi}{4}$

B. $\frac{\pi}{2}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{6}$

Answer: C

Solution:

Solution:
Given, $y = 3\sin\theta \, . \cos\theta$
$\frac{dy}{d\theta} = 3[\sin\theta(-\sin\theta) + \cos\theta(\cos\theta)]$
$\frac{dy}{d\theta} = 3[\cos^2\theta - \sin^2\theta] \; = 3\cos 2\theta$ ......(i)
and $x = e^{\theta}\sin\theta$
$\frac{dx}{d\theta} = e^{\theta}\cos\theta + \sin\theta\, e^{\theta}$
$\frac{dx}{d\theta} = e^{\theta}(\sin\theta + \cos\theta)$ ......(ii)
Dividing (i) by (i)
$\frac{dy}{dx} = \frac{3\cos 2\theta}{e^{\theta}(\sin\theta + \cos\theta)} = \frac{3(\cos^2\theta - \sin^2\theta)}{e^{\theta}(\sin\theta + \cos\theta)}$
$\frac{dy}{dx} = \frac{3(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}{e^{\theta}(\sin\theta + \cos\theta)}$
$\frac{dy}{dx} = \frac{3(\cos\theta - \sin\theta)}{e^{\theta}}$
Given tangent is parallel to x -axis then $\frac{dy}{dx} = 0$
$0 = \frac{3(\cos\theta - \sin\theta)}{e^{\theta}}$
or $\cos\theta - \sin\theta = 0 \Rightarrow \cos\theta = \sin\theta$
$\Rightarrow \tan\theta = 1 \Rightarrow \tan\theta = \frac{\tan\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$

Question183

If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha\log|x| + \beta x^2 + x$ then

Options:

A. $\alpha = 2, \beta = -\frac{1}{2}$

B. $\alpha = 2, \beta = \frac{1}{2}$

C. $\alpha = -6, \beta = \frac{1}{2}$

D. $\alpha = -6, \beta = -\frac{1}{2}$

Answer: A

Solution:

Solution:
Let $f(x) = \alpha \log|x| + \beta x^2 + x$
Differentiate both side,
$f'(x) = \alpha x + 2\beta x + 1$
Since $x = -1$ and $x = 2$ are extreme points therefore $f'(x) = 0$ at these points.
Put $x = -1$ and $x = 2$ in $f'(x)$, we get
$-\alpha - 2\beta + 1 = 0 \Rightarrow \alpha + 2\beta = 1$ ......(i)
$\alpha 2 + 4\beta + 1 = 0 \Rightarrow \alpha + 8\beta = -2$ ....(ii)
On solving (i) and (ii), we get
$6\beta = -3 \Rightarrow \beta = -\frac{1}{2}$
$\therefore \alpha = 2$

Question184

The minimum area of a triangle formed by any tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{81} = 1$ and the co-ordinate axes is:
[Online April 12, 2014]

Options:

A. 12

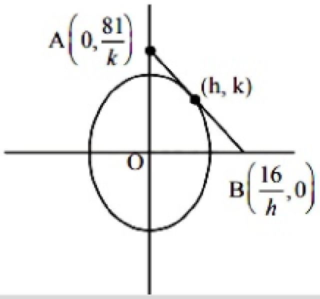
B. 18

C. 26

D. 36

Answer: D

Solution:



Let (h, k) be the point on ellipse through which tangent is passing.

Equation of tangent at (h, k) = $\frac{xh}{16} + \frac{yk}{81} = 1$

at $y = 0, x = \frac{16}{h}$

at $x = 0, y = \frac{81}{k}$

Area of AOB = $\frac{1}{2} \times \left(\frac{16}{h}\right) \times \left(\frac{81}{k}\right) = \frac{648}{hk}$

$A^2 = \frac{(648)^2}{h^2k^2}$ ....(i)

(h, k) must satisfy equation of ellipse

$\frac{h^2}{16} + \frac{k^2}{81} = 1$

$h^2 = \frac{16}{81}(81 - k^2)$

Putting value of h^2 in equation (i)

$A^2 = \frac{81(648)^2}{16 \times k^2(81 - k^2)} = \frac{\alpha}{81k^2 - k^4}$

differentiating w.r. to k

$2AA' = \alpha\left(\frac{-1}{81k^2 - k^4}\right)(162k - 4k^3)$

$2AA' = -2A(81k - 4k^3) \Rightarrow A' = -81k - 4k^3$

Put $A' = 0$

$\Rightarrow 162k - 4k^3 = 0, k(162 - 4k^2) = 0$

$\Rightarrow k = 0, k = \pm\frac{9}{\sqrt{2}}$

$A'' = -(81 - 12k^2)$

For both value of k, $A'' = 405 > 0$

Area will be minimum for $k = \pm\frac{9}{\sqrt{2}}$

$h^2 = \frac{16}{81}(81 - k^2) = 8$

$h = \pm 2\sqrt{2}$

Area of triangle AOB = $\frac{648 \times \sqrt{2}}{2\sqrt{2} \times 9}$ = 36 sq unit

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Question185

The volume of the largest possible right circular cylinder that can be inscribed in a sphere of radius $= \sqrt{3}$ is:
[Online April 11, 2014]

Options:

A. $\frac{4}{3}\sqrt{3}\pi$

B. $\frac{8}{3}\sqrt{3}\pi$

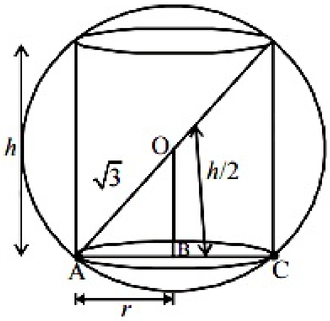
C. 4π

D. 2π

Answer: C

Solution:

Given, radius of sphere $= \sqrt{3}$
Now, In ΔOAB, by Pythagoras theorem
$(OA)^2 = (OB)^2 + (AB)^2$



$(\sqrt{3})^2 = \left(\frac{h}{2}\right)^2 + r^2$

$3 = \frac{h^2}{4} + r^2 \Rightarrow r^2 = 3 - \frac{h^2}{4}$ ......(i)

Now, volume of cylinder $= \pi r^2 h$

$V = \pi\left(3 - \frac{h^2}{4}\right) h$ (using eq. (i))

$V = 3\pi h - \frac{\pi h^3}{4}$ ......(ii)

Now, for largest possible right circular cylinder the volume must be maximum

$\therefore$ For maximum volume, $\frac{dV}{dh} = 0$

Now, Differentiating eq. (2) w.r.t. h

$\frac{dV}{dh} = 3\pi - \frac{3}{4}\pi h^2$

or $3\pi - \frac{3}{4}\pi h^2 = 0 \Rightarrow 3\pi = \frac{3}{4}\pi h^2$

$\Rightarrow h^2 = 4 \Rightarrow h = 2$

Now, volume (V) of the cylinder $= \pi\left(3 - \frac{h^2}{4}\right) h = \pi(6 - 2) = 4\pi$

Question186

A spherical balloon is being inflated at the rate of 35cc/min. The rate of increase in the surface area (in cm^2/min.) of the balloon when its diameter is 14 cm, is :
[Online April 25, 2013]

Options:

A. 10

B. $\sqrt{10}$

C. 100

D. $10\sqrt{10}$

Answer: A

Solution:

Solution:

Volume of sphere $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3} . \pi . 3r^2 . \frac{dr}{dt}$

$35 = 4\pi r^2 . \frac{dr}{dt}$ or $\frac{dr}{dt} = \frac{35}{4\pi r^2}$ ....(i)

Surface area of sphere $= S = 4\pi r^2$

$\frac{dS}{dt} = 4\pi \times 2r \times \frac{dr}{dt} = 8\pi r . \frac{dr}{dt}$

$\frac{dS}{dt} = \frac{70}{r}$ (By using (i))

Now, diameter $= 14cm, r = 7$

$\therefore \frac{dS}{dt} = 10$

Question187

If the surface area of a sphere of radius r is increasing uniformly at the rate $8cm^2$ / s, then the rate of change of its volume is :
[Online April 9, 2013]

Options:

A. constant

B. proportional to $\sqrt{r}$

C. proportional to r^2

D. proportional to r

Answer: D

Solution:

Solution:

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 . \frac{dr}{dt}$ ......(i)

$S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r . \frac{dr}{dt}$

$\Rightarrow 8 = 8\pi r \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi r}$

Putting the value of $\frac{dr}{dt}$ in (i), we get

$\frac{dV}{dt} = 4\pi r^2 \times \frac{1}{\pi r} = 4r$

$\Rightarrow \frac{dV}{dt}$ is proportional to r.

Question188

The real number k for which the equation, $2x^3 + 3x + k = 0$ has two

[2013]

Options:

A. lies between 1 and 2

B. lies between 2 and 3

C. lies between .1 and 0

D. does not exist.

Answer: D

Solution:

Solution:
$f(x) = 2x^3 + 3x + k$
$f'(x) = 6x^2 + 3 > 0 \ \forall x \in R (\because x^2 > 0)$
$\Rightarrow f(x)$ is strictly increasing function
$\Rightarrow f(x) = 0$ has only one real root, so two roots are not possible.

Question189

Statement-1: The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$. Statement-2: The functions x^2e^x and x^2e^{-x} are increasing for all $x > 0$ and the sum of two increasing functions in any interval (a, b) is an increasing function in (a, b). [Online April 22, 2013]

Options:

A. Statement-1 is false; Statement-2 is true.

B. Statement-1is true; Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

C. Statement-1 is true; Statement-2 is false.

D. Statement-1is true; Statement-2 is true; Statement-2 is a correct explanation for statement-1.

Answer: C

Solution:

Solution:
Let $y = x^2 . e^{-x}$
For increasing function,
$\frac{dy}{dx} > 0 \Rightarrow x[(2 - x)e^{-x}] > 0$
$\because x > 0, \therefore (2 - x)e^{-x} > 0$
$\Rightarrow (2 - x)\frac{1}{e^x} > 0$
For $0 < x < 2$, $(2 - x) < 0$
$\therefore \frac{1}{e^x} < 0$, but it is not possible

Question190

Statement-1: The equation x log x = 2 - x is satisfied by at least one value of x lying between 1 and 2.
Statement-2: The function f (x) = x log x is an increasing function in [1, 2] and g (x) = 2 - x is a decreasing function in [1, 2] and the graphs represented by these functions intersect at a point in [1, 2]
[Online April 9, 2013]

Options:

A. Statement-1 is true; Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

B. Statement-1 is true; Statement-2 is true; Statement-2 is not correct explanation for Statement-1.

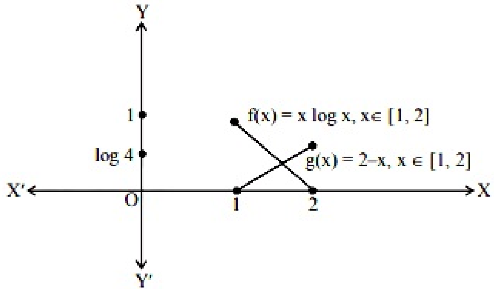
C. Statement-1 is false, Statement-2 is true.

D. Statement-1 is true, Statement-2 is false.

Answer: A

Solution:

Solution:
$f(x) = x \log x,\ f(1) = 0,\ f(2) = 4$
$g(x) = 2 - x,\ g(1) = 1,\ g(2) = 0$
$\log 10 > \log 4 \Rightarrow 1 > \log 4$



Thus statement -1 and 2 both are true and statement-2 is a correct explanation of statement 1.

Question191

If an equation of a tangent to the curve, y − cos(x + f), −1 − 1 ≤ x ≤ 1 + π, is x + 2y = k then k is equal to :
[Online April 25, 2013]

Options:

A. l

B. 2

C. $\frac{\pi}{}$

D. $\frac{\pi}{2}$

Answer: D

Solution:

Solution:
Let $y = \cos(x + y)$
$\Rightarrow \frac{dy}{dx} = -\sin(x + y)\left(1 + \frac{dy}{dx}\right)$ ......(i)
Now, given equation of tangent is
$x + 2y = k$
$\Rightarrow$ Slope $= \frac{-1}{2}$
So, $\frac{dy}{dx} = \frac{-1}{2}$ put this value in (i), we get
$\frac{-1}{2} = -\sin(x + y)\left(1 - \frac{1}{2}\right)$
$\Rightarrow \sin(x + y) = 1$
$\Rightarrow x + y = \frac{\pi}{2} \Rightarrow y = \frac{\pi}{2} - x$
Now, $\frac{\pi}{2} - x = \cos(x + y)$
$\Rightarrow x = \frac{\pi}{2}$ and $y = 0$
Thus $x + 2y = k \Rightarrow \frac{\pi}{2} = k$

Question192

The cost of running a bus from A to B, is Rs. $\left(av + \frac{b}{v}\right)$, where v km / h is the average speed of the bus. When the bus travels at 30km / h, the cost comes out to be Rs.75 while at 40km / h, it is Rs. 65. Then the most economical speed (in km / h) of the bus is:
[Online April 23, 2013]

Options:

A. 45

B. 50

C. 60

D. 40

Answer: C

Solution:

Solution:
Let cost $C = av + \frac{b}{v}$
According to given question,
$30a + \frac{b}{30} = 75$ ....(i)

On solving (i) and (ii), we get

$a = \frac{1}{2}$ and $b = 1800$

Now, $C = av + \frac{b}{v}$

$\Rightarrow \frac{dC}{dv} = a - \frac{b}{v^2}$

$\frac{dC}{dv} = 0 \Rightarrow a - \frac{b}{v^2} = 0$

$\Rightarrow v = \sqrt{\frac{b}{a}} = \sqrt{3600} \Rightarrow v = 60\,\text{kmph}$

Question193

The maximum area of a right angled triangle with hypotenuse h is [Online April 22, 2013]

Options:

A. $\frac{h^2}{2\sqrt{2}}$

B. $\frac{h^2}{2}$

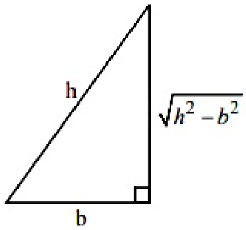
C. $\frac{h^2}{\sqrt{2}}$

D. $\frac{h^2}{4}$

Answer: D

Solution:

Solution:
Let base = b



Altitude (or perpendicular) $= \sqrt{h^2 - b^2}$

Area, $A = \frac{1}{2} \times$ base $\times$ altitude $= \frac{1}{2} \times b \times \sqrt{h^2 - b^2}$

$\Rightarrow \frac{dA}{db} = \frac{1}{2}\left[\sqrt{h^2 - b^2} + b.\frac{-2b}{2\sqrt{h^2 - b^2}}\right]$

$= \frac{1}{2}\left[\frac{h^2 - 2b^2}{\sqrt{h^2 - b^2}}\right]$

Put $\frac{dA}{db} = 0, \Rightarrow b = \frac{h}{\sqrt{2}}$

Maximum area $= \frac{1}{2} \times \frac{h}{\sqrt{2}} \times \sqrt{h^2 - \frac{h^2}{2}} = \frac{h^2}{4}$

Question194

A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is: [2012]

Options:

A. $\frac{9}{7}$

B. $\frac{7}{9}$

C. $\frac{2}{9}$

D. $\frac{9}{2}$

Answer: C

Solution:

Solution:

Volume of spherical balloon $= V = \frac{4}{3}\pi r^3$

Differentiate both the side, w.r.t 't' we get,

$\frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$

∴ After 49min

Volume $= (4500 - 49 \times 72)\pi = (4500 - 3528)\pi = 972\pi m^3$

$\Rightarrow V = 972\pi m^3$

$\therefore 972\pi = \frac{4}{3}\pi r^3$

$\Rightarrow r^3 = 3 \times 243 = 3 \times 3^5 = 3^6 = (3^2)^3$

$\Rightarrow r = 9$

Given $\frac{dV}{dt} = 72\pi$

Putting $\frac{dV}{dt} = 72\pi$ and $r = 9$, we get

$\therefore 72\pi = 4\pi \times 9 \times 9\left(\frac{dr}{dt}\right)$

$\Rightarrow \frac{dr}{dt} = \left(\frac{2}{9}\right)$

Question195

If a metallic circular plate of radius 50cm is heated so that its radius increases at the rate of 1mm per hour, then the rate at which, the area of the plate increases (in cm^2/ hour) is [Online May 26, 2012]

Options:

A. 5π

B. 10π

C. 100π

D. 50π

Answer: B

Solution:

Let $A = \pi r^2$ be area of metalic circular plate of $r = 50\text{cm}$.
Also, given $\frac{dr}{dt} = 1\text{mm} = \frac{1}{10}\text{cm}$
$\therefore A = \pi r^2$
$\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi \,.\, 50 \,.\, \frac{1}{10} = 10\pi$
Hence, area of plate increases in $10\pi\ \text{cm}^2/\text{hour}$.

Question196

The weight W of a certain stock of fish is given by $W = nw$, where n is the size of stock and w is the average weight of a fish. If n and w change with time t as $n = 2t^2 + 3$ and $w = t^2 - t + 2$, then the rate of change of W with respect to tat $t = 1$ is
[Online May 19, 2012]

Options:

A. 1

B. 8

C. 13

D. 5

Answer: C

Solution:

Solution:
Let $W = nw$
$\Rightarrow \frac{dW}{dt} = n\frac{dw}{dt} + w \,.\, \frac{dn}{dt}$ ......(i)
Given : $w = t^2 - t + 2$ and $n = 2t^2 + 3$
$\Rightarrow \frac{dw}{dt} = 2t - 1$ and $\frac{dn}{dt} = 4t$
$\therefore$ Equation(i)
$\Rightarrow \frac{dw}{dt} = (2t^2 + 3)(2t - 1) + (t^2 - t + 2)(4t)$
Thus, $\left.\frac{dW}{dt}\right|_{t=1} = (2 + 3)(2 - 1) + (2)$
$= 5(1) + 8 = 13$

Question197

Consider a rectangle whose length is increasing at the uniform rate of 2 m/sec, breadth is decreasing at the uniform rate of 3 m/sec and the area is decreasing at the uniform rate of 5 m^2/sec. If after some time the breadth of the rectangle is 2 m then the length of the rectangle is [Online May 12, 2012]

Options:

A. 2 m

B. 4 m

C. 1 m

D. 3 m

Answer: D

Solution:

Solution:
Let A be the area ,b be the breadth and ell be the length of the rectangle.
Given : $\frac{dA}{dt} = -5, \frac{dl}{dt} = 2, \frac{db}{dt} = -3$
We know, $A = l \times b$
$\Rightarrow \frac{dA}{dt} = l . \frac{db}{dt} + b . \frac{dl}{dt} = -3l + 2b$
$\Rightarrow -5 = -3l + 2b$
When $b = 2$, we have
$-5 = -3l + 4 \Rightarrow l = \frac{9}{3} = 3m$

Question198

If a circular iron sheet of radius 30cm is heated such that its area increases at the uniform rate of $6\pi cm^2$ / hr, then the rate (in mm / hr) at which the radius of the circular sheet increases is [Online May 7, 2012]

Options:

A. 1.0

B. 0.1

C. 1.1

D. 2.0

Answer: B

Solution:

Let $A = \pi r^2$.

$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

$6\pi = 2\pi(30) \cdot \frac{dr}{dt}$

$\Rightarrow \frac{3}{30} = \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{10} = 0.1$

Thus, the rate at which the radius of the circular sheet increases is 0.1

Question199

If $f(x) = xe^{x(1-x)}$, $x \in R$, then f (x) is [Online May 12, 2012]

Options:

A. decreasing on [–1/2, 1]

B. decreasing on R

C. increasing on [–1/2, 1]

D. increasing on R

Answer: C

Solution:

Solution:

$f(x) = xe^{x(1-x)}, x \in R$

$f'(x) = e^{x(1-x)} \cdot [1 + x - 2x^2]$

$= -e^{x(1-x)} \cdot [2x^2 - x - 1]$

$= -2e^{x(1-x)} \cdot \left[\left(x + \frac{1}{2}\right)(x - 1)\right]$

$f'(x) = -2e^{x(1-x)} \cdot A$

where $A = \left(x + \frac{1}{2}\right)(x - 1)$

Now, exponential function is always +ve and $f'(x)$ will be opposite to the sign of A which is − ve in $\left[-\frac{1}{2}, 1\right]$

Hence, $f'(x)$ is +ve in $\left[-\frac{1}{2}, 1\right]$

$\therefore f(x)$ is increasing on $\left[-\frac{1}{2}, 1\right]$

Question200

The equation of the normal to the parabola, $x^2 = 8y$ at $x = 4$ is [Online May 19, 2012]

Options:

A. $x + 2y = 0$

B. $x + y = 2$

D. $x + y = 6$

Answer: D

Solution:

Solution:
$x^2 = 8y$ .....(i)
When, $x = 4$, then $y = 2$
Now $\frac{dy}{dx} = \frac{2x}{8} = \frac{x}{4}, \left.\frac{dy}{dx}\right]_{x=4} = 1$
Slope of normal $= -\frac{1}{\frac{dy}{dx}} = -1$
Euqation of normal at $x = 4$ is
$y - 2 = -1(x - 4)$
$\Rightarrow y = -x + 4 + 2 = -x + 6$
$\Rightarrow x + y = 6$

Question201

Let $a, b \in R$ be such that the function f given by $f(x) = \ln|x| + bx^2 + ax, x \neq 0$ has extreme values at $x = -1$ and $x = 2$
Statement-1 : f has local maximum at $x = -1$ and at $x = 2$.
Statement- 2: $a = \frac{1}{2}$ and $b = \frac{-1}{4}$
[2012]

Options:

A. Statement-1 is false, Statement-2 is true.

B. Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.

C. Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for Statement-1.

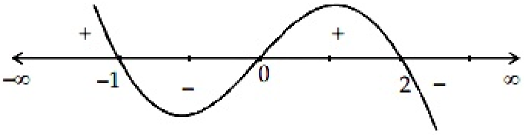
D. Statement-1 is true, statement-2 is false.

Answer: B

Solution:

Solution:
Given that, $f(x) = \ln|x| + bx^2 + ax$
$\therefore f'(x) = \frac{1}{x} + 2bx + a$
At $x = -1$, $f'(x) = -1 - 2b + a = 0$
$\Rightarrow a - 2b = 1$ .....(i)
At $x = 2$, $f'(x) = \frac{1}{2} + 4b + a = 0$
$\Rightarrow a + 4b = -\frac{1}{2}$ ......(ii)
On solving (i) and(ii) we get $a = \frac{1}{2}, b = -\frac{1}{4}$
Thus, $f'(x) = \frac{1}{x} - \frac{x}{2} + \frac{1}{2} = \frac{2 - x^2 + x}{2x}$

$= \frac{-x^2 + x + 2}{2x} = \frac{-(x^2 - x - 2)}{2x} = \frac{-(x+1)(x-2)}{2x}$



So maxima at $x = -1, 2$

Question202

A line is drawn through the point (1,2) to meet the coordinate axes at P and Q such that it forms a triangle OPQ, where O is the origin. If the area of the triangle OPQ is least, then the slope of the line PQ is : [2012]

Options:

A. $-\frac{1}{4}$

B. -4

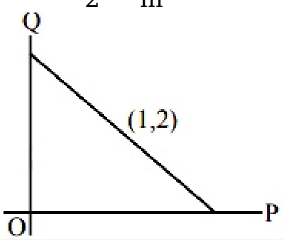
C. -2

D. $-\frac{1}{2}$

Answer: C

Solution:

Solution:
Equation of a line passing through (x_1, y_1) having slope m is given by $y - y_1 = m(x - x_1)$
Since the line PQ is passing through (1,2) therefore its equation is $(y - 2) = m(x - 1)$
where m is the slope of the line PQ.
Now, point $P(x, 0)$ will also satisfy the equation of PQ
$\therefore y - 2 = m(x - 1) \Rightarrow 0 - 2 = m(x - 1)$
$\Rightarrow -2 = m(x - 1) \Rightarrow x - 1 = \frac{-2}{m}$
$\Rightarrow x = \frac{-2}{m} + 1$
Also, $OP = \sqrt{(x - 0)^2 + (0 - 0)^2} = x = \frac{-2}{m} + 1$
Similarly, point $Q(0, y)$ will satisfy equation of PQ
$\therefore y - 2 = m(x - 1)$
$\Rightarrow y - 2 = m(-1)$
$\Rightarrow y = 2 - m$ and $OQ = y = 2 - m$
Area of $\Delta POQ = \frac{1}{2}(OP)(OQ) = \frac{1}{2}\left(1 - \frac{2}{m}\right)(2 - m)$ ($\because$ Area of $\Delta = \frac{1}{2} \times$ base $\times$ height)
$= \frac{1}{2}\left[2 - m - \frac{4}{m} + 2\right] = \frac{1}{2}\left[4 - \left(m + \frac{4}{m}\right)\right]$
$= 2 - \frac{m}{2} - \frac{2}{m}$



Now, $f'(m) = \frac{-1}{2} + \frac{2}{m^2}$

Put $f'(m) = 0$

$\Rightarrow m^2 = 4 \Rightarrow m = \pm 2$

Now, $f''(m) = \frac{-4}{m^3}$

$f''(m)|_{m=2} = -\frac{1}{2} < 0$

$f''(m)|_{m=-2} = \frac{1}{2} > 0$

Area will be least at $m = -2$

Hence, slope of PQ is -2 .

Question203

Let $f : (-\infty, \infty) \to (-\infty, \infty)$ be defined by $f(x) = x^3 + 1$.

Statement 1 : The function f has a local extremum at $x = 0$

Statement 2: The function f is continuous and differentiable on $(-\infty, \infty)$ and $f'(0) = 0$

[Online May 26, 2012]

Options:

A. Statement 1 is true, Statement 2 is false.

B. Statement 1 is true, Statement 2 is true, Statement 2 is a correct explanation for Statement 1.

C. Statement 1 is true, Statement 2 is true, Statement 2 is not the correct explanation for Statement 1.

D. Statement 1 is false, Statement 2 is true.

Answer: D

Solution:

Solution:

Let $f : (-\infty, \infty) \to (-\infty, \infty)$ be defined by $f(x) = x^3 + 1$.

Clearly, $f(x)$ is symmetric along $y = 1$ and it has neither maxima nor minima.

therefore Statement- 1 is false.

Hence, option (d) is correct.

Question204

Let f be a function defined by -

$$f(x) = \begin{cases} \frac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Statement -1: $x = 0$ is point of minima of f

Statement -2: $f'(0) = 0$

Options:

A. Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for statement-1.

B. Statement-1 is true, statement-2 is true; statement-2 is NOT a correct explanation for statement-1.

C. Statement-1 is true, statement-2 is false.

D. Statement-1 is false, statement-2 is true.

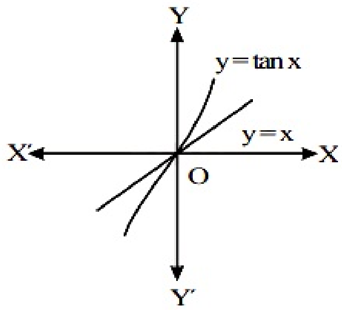
Answer: B

Solution:

Solution:

$$f(x) = \begin{cases} \frac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

For $x > 0$
$\tan x > x$
$\frac{\tan x}{x} > 1$



For $x<0 \⇒\tan x ⇒ $\frac{\tan x}{x} > 1$
$f(0) = 1$ at $x = 0$
$\Rightarrow x = 0$ is the point of minima
So, Statement 1 is true. Statement 2 is also true.

Question205

For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, d\, t$. Then f has [2011]

Options:

A. local minimum at π and 2π

B. local minimum at π and local maximum at 2π

C. local maximum at π and local minimum at 2π

D. local maximumat π and 2π

Answer: C

Solution:

Solution:
$f'(x) = \sqrt{x}\sin x$
$f'(x) = 0$
$\Rightarrow x = 0$ or $\sin x = 0$
$\Rightarrow x = 2\pi, \pi$
$f''(x) = \sqrt{x}\cos x + \frac{1}{2\sqrt{x}}\sin x$
$= \frac{1}{2\sqrt{x}}(2x\cos x + \sin x)$
At $x = \pi$, $f''(x) < 0$
Hence, local maxima at $x = \pi$
At $x = 2\pi$, $f''(x) > 0$
Hence local minima at $x = 2\pi$

Question206

The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x -axis, is
[2010]

Options:

A. $y = 1$

B. $y = 2$

C. $y = 3$

D. $y = 0$

Answer: C

Solution:

Solution:
Since the tangent is parallel to x -axis,
$\therefore \frac{dy}{dx} = 0 \Rightarrow 1 - \frac{8}{x^3} = 0 \Rightarrow x = 2 \Rightarrow y = 3$
Equation of the tangent is $y - 3 = 0(x - 2)$
$\Rightarrow y = 3$

Question207

Let $f : R \rightarrow R$ be a continuous function defined by
$f(x) = \frac{1}{e^x + 2e^{-x}}$
Statement -1: $f(c) = \frac{1}{3}$, for some $c \in R$
Statement -2: $0 < f(x) \leq \frac{1}{2\sqrt{2}}$, for all $x \in R$
[2010]

Options:

A. Statement -1 is true, Statement -2 is true ; Statement - 2 is not a correct explanation for Statement -1

B. Statement -1 is true, Statement -2 is false.

C. Statement -1 is false, Statement -2 is true .

D. Statement - 1 is true, Statement 2 is true ; Statement -2 is a correct explanation for Statement -1.

Answer: D

Solution:

Solution:

Given $f(x) = \frac{1}{e^x + 2e^{-x}} = \frac{e^x}{e^{2x} + 2}$

$f'(x) = \frac{(e^{2x} + 2)e^x - 2e^{2x} . e^x}{(e^{2x} + 2)^2}$

$f'(x) = 0 \Rightarrow e^{2x} + 2 = 2e^{2x}$

$\Rightarrow e^{2x} = 2 \Rightarrow e^x = \sqrt{2}$

$\because f''(\sqrt{2}) = +ve$

$\therefore$ Maximum values of $f(x) = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$

$\Rightarrow 0 < f(x) \leq \frac{1}{2\sqrt{2}} \ \forall x \in R$

Since, $0 < \frac{1}{3} < \frac{1}{2\sqrt{2}}$

$\Rightarrow$ for some $c \in R$, $f(c) = \frac{1}{3}$

Question208

Let f : R → R be defined by

$$f(x) = \begin{cases} k - 2x, & \text{if } x \leq -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$

If f has a local minimum at $x = -1$, then a possible value of k is [2010]

Options:

A. 0

B. $-\frac{1}{2}$

C. -1

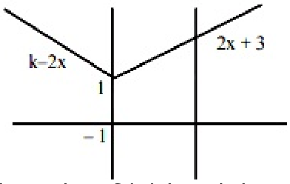
D. 1

Answer: C

Solution:

Solution:

$f(x) = \begin{cases} k - 2x, & \text{if } x \leq -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$



Clear that f (x) is minimum at (-1,1)
$\therefore f(-1) = 1$
$1 = k + 2 \Rightarrow k = -1$

Question209

For real x, let $f(x) = x^3 + 5x + 1$, then [2009]

Options:

A. f is onto R but not one-one

B. f is one-one and onto R

C. f is neither one-one nor onto R

D. f is one-one but not onto R

Answer: B

Solution:

Solution:
Given that $f(x) = x^3 + 5x + 1$
$\therefore f'(x) = 3x^2 + 5 > 0, \forall x \in R$
$\Rightarrow f(x)$ is strictly increasing on R
$\Rightarrow f(x)$ is one one
$\therefore$ Being a polynomial $f(x)$ is continuous and increasing.
on R with $\lim_{x \to \infty} f(x) = -\infty$
and $\lim_{x \to \infty} f(x) = \infty$
$\therefore$ Range of $f = (-\infty, \infty) = R$
Hence f is onto also. So, f is one one and onto R.

Question210

Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that x = 0 is the onlyreal root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval [-1,1]: [2009]

Options:

A. $P(-1)$ is not minimum but $P(1)$ is the maximum of P

B. $P(-1)$ is the minimum but $P(1)$ is not the maximum of P

C. Neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

Answer: A

Solution:

Solution:
Given that $P(x) = x^4 + ax^3 + bx^2 + cx + d$
$\Rightarrow P'(x) = 4x^3 + 3ax^2 + 2bx + c$
But given $P'(0) = 0 \Rightarrow c = 0$
$\therefore P(x) = x^4 + ax^3 + bx^2 + d$
Again given that $P(-1) < P(1)$
$\Rightarrow 1 - a + b + d < 1 + a + b + d$
$\Rightarrow a > 0$
Now $P'(x) = 4x^3 + 3ax^2 + 2bx = x(4x^2 + 3ax + 2b)$
As $P'(x) = 0$, there is only one solution $x = 0$, therefore $4x^2 + 3ax + 2b = 0$ should not have any real roots i.e. $D < 0$
$\Rightarrow 9a^2 - 32b < 0 \Rightarrow b > \frac{9a^2}{32} > 0$
Hence $a, b > 0$
$\Rightarrow P'(x) = 4x^3 + 3ax^2 + 2bx > 0 \ \forall x > 0$
$\therefore P(x)$ is an increasing function on (0,1)
$\therefore P(0) < P(a)$
Similarly we can prove $P(x)$ is decreasing on (-1,0)
$\therefore P(-1) > P(0)$
So we can conclude that
$\text{Max}\, P(x) = P(1)$ and $\text{Min}\, P(x) = P(0)$
$\Rightarrow P(-1)$ is not minimum but $P(1)$ is the maximum of P.

Question211

How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have? [2008]

Options:

A. 7

B. 1

C. 3

D. 5

Answer: B

Solution:

Solution:
Let $f(x) = x^7 + 14x^5 + 16x^3 + 30x - 560$
$\Rightarrow f'(x) = 7x^6 + 70x^4 + 48x^2 + 30 > 0, \ \forall x \in R$ ......(i)
$\Rightarrow f$ is an increasing function on R
Also $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$ ....(ii)
From (i) and (ii) clear that the curve
$y = f(x)$ crosses x -axis only once.
$\therefore f(x) = 0$ has exactly one real root.

Suppose the cubic $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds?
[2008]

Options:

A. The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$

B. The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

C. The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

D. The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

Answer: A

Solution:

Solution:

Let $y = x^3 - px + q \Rightarrow \frac{dy}{dx} = 3x^2 - p$

For maxima and minima

$\frac{dy}{dx} = 0 \Rightarrow 3x^2 - p = 0 \Rightarrow x = \pm\sqrt{\frac{p}{3}}$

$\frac{d^2y}{dx^2} = 6x \frac{d^2y}{dx^2}\Big|_{x = \sqrt{\frac{p}{3}}} = +\text{ve}$ and $\frac{d^2y}{dx^2}\Big|_{x = -\sqrt{\frac{p}{3}}} = -\text{ve}$

$\therefore$ y has minimum at $x = \sqrt{\frac{p}{3}}$ and maximum at

$x = -\sqrt{\frac{p}{3}}$

Question213

The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in
[2007]

Options:

A. $\left(0, \frac{\pi}{2}\right)$

B. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

C. $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

D. $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

Solution:

Solution:
Given that $f(x) = \tan^{-1}(\sin x + \cos x)$
Differentiate w.r.t. to x

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2}.(\cos x - \sin x)$$

$$= \frac{\sqrt{2}.\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)}{1 + (\sin x + \cos x)^2}$$

$$= \frac{\sqrt{2}\left(\cos\frac{\pi}{4}.\cos x - \sin\frac{\pi}{4}.\sin x\right)}{1 + (\sin x + \cos x)^2}$$

$$\therefore f'(x) = \frac{\sqrt{2}\cos\left(x + \frac{\pi}{4}\right)}{1 + (\sin x + \cos x)^2}$$

Given that f (x) is increasing

$$\therefore f'(x) > 0 \Rightarrow \cos\left(x + \frac{\pi}{4}\right) > 0$$

$$\Rightarrow -\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2}$$

$$\Rightarrow -\frac{3\pi}{4} < x < \frac{\pi}{4}$$

Hence, f (x) is increasing when

$$n \in \left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$$

Question214

Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2,0) and (3,0) is [2006]

Options:

A. π

B. $\frac{\pi}{2}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{4}$

Answer: B

Solution:

Solution:
$\frac{dy}{dx} = 2x - 5 \therefore m_1 = (2x - 5)_{(2,\,0)} = -1$
$m_2 = (2x - 5)_{(3,\,0)} = 1 \Rightarrow m_1 m_2 = -1$
i.e. the tangents are perpendicular to each other.

The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at [2006]

Options:

A. $x = 2$

B. $x = -2$

C. $x = 0$

D. $x = 1$

Answer: A

Solution:

Solution:

Given $f(x) = \frac{x}{2} + \frac{2}{x} \Rightarrow f'(x) = \frac{1}{2} - \frac{2}{x^2} = 0$

$\Rightarrow x^2 = 4 \Rightarrow x = 2, -2;$

Now, $f''(x) = \frac{4}{x^3}$

$f''(x)]_{x=2} = +ve \Rightarrow f(x)$ has local min at $x = 2$

Question216

Two points A and B move from rest along a straight line with constant acceleration f and f' respectively. If A takes m sec. more than B and describes 'n'units more than B in acquiring the same speed then [2005]

Options:

A. $(f - f')m^2 = ff'n$

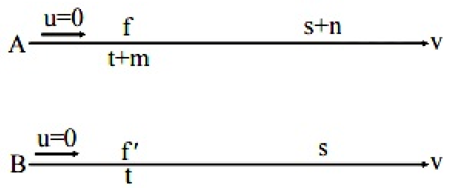
B. $(f + f')m^2 = ff'n$

C. $\frac{1}{2}(f + f')m = ff'n^2$

D. $(f' - f)n = \frac{1}{2}ff'm^2$

Answer: D

Solution:



As per question if point B moves s distance in t time then point A moves (s + n) distance in time (t + m) after which both have same velocity v.

Then using equation v = u + at we get

$v = f(t+m) = f't \Rightarrow t = \frac{fm}{f'-f}$ ......(i)

Using equation $v^2 = u^2 + 2$, as we get

$v^2 = 2f(s+n) = 2f's \Rightarrow s = \frac{fn}{f'-f}$ ....(ii)

Also for point B using the eqn $s = ut + \frac{1}{2}at^2$, we get

$s = \frac{1}{2}f't^2$

Substituting values of t and s from equations (i) and (ii) in the above relation, we get

$\frac{fn}{f'-f} = \frac{1}{2}f'\frac{f^2m^2}{(f'-f)^2}$

$\Rightarrow (f'-f)n = \frac{1}{2}ff'm^2$

Question217

A lizard, at an initial distance of 21 cm behind an insect, moves from rest with an acceleration of 2cm / s^2 pursues the insect which is crawling uniformly along a straight line at a speed of 20 cm/s. Then the liard will catch the insect after
[2005]

Options:

A. 20 s

B. 1 s

C. 21 s

D. 24 s

Answer: C

Solution:

Solution:
Let the liard catches the insect after time t then distance covered by liard = 21cm + distance covered by insect

$\Rightarrow \frac{1}{2}ft^2 = 4 \times t + 21$

$\Rightarrow \frac{1}{2} \times 2 \times t^2 = 20 \times t + 21$

$\Rightarrow t^2 - 20t - 21 = 0 \Rightarrow t = 21sec$

Question218

A spherical iron ball 10cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50cm^3 / min. When the thickness of ice is 5cm,then the rate at which the thickness of ice decreases is
[2005]

A. $\frac{1}{36\pi}$cm/min

B. $\frac{1}{18\pi}$cm/min

C. $\frac{1}{54\pi}$cm/min

D. $\frac{5}{6\pi}$cm/min

Answer: B

Solution:

Solution:
Given that Total radius r = 10 + 5 = 15cm

$\frac{dv}{dt} = 50cm^3 / min \Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 50$

$\Rightarrow 4\pi r^2 \frac{dr}{dt} = 50$

$\Rightarrow \frac{dr}{dt} = \frac{50}{4\pi(15)^2} = \frac{1}{18\pi} cm / min$

Question219

A function is matched below against an interval where it is supposed to be increasing. Which of the following pairs is incorrectly matched? [2005]

Options:

A.

Interval	Function
$(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$

B.

Interval	Function
$[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$

C.

Interval	Function
$\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$

D.

Interval	Function
$(-\infty, -4)$	x^3+6x^2+6

Answer: C

Solution:

Solution:
From option(c) ,$f(x) = 3x^2 - 2x + 1$ is increasing
when $f'(x) = 6x - 2 \ge 0$
$\Rightarrow x \in [1/3, \infty)$
$\therefore f(x)$ is incorrectly matched with $\left(-\infty, \frac{1}{3}\right]$

Question220

The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point θ is such that [2005]

Options:

A. it passes through the origin

B. it makes an angle frac pi 2 + theta with the x -axis

C. it passes through left(a frac pi 2, −a right)

D. It is at a constant distance from the origin

Answer: D

Solution:

Solution:
Given $x = a(\cos\theta + \theta\sin\theta)$
$\Rightarrow \frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)$
$\Rightarrow \frac{dx}{d\theta} = a\theta\cos\theta$ ....(i)
$y = a(\sin\theta - \theta\cos\theta)$
$\frac{dy}{d\theta} = a[\cos\theta - \cos\theta + \theta\sin\theta]$
$\Rightarrow \frac{dy}{d\theta} = a\theta\sin\theta$ ......(ii)
From equations (i) and (ii) we get
$\frac{dy}{dx} = \tan\theta \Rightarrow$ Slope of normal $= -\cot\theta$
Equation of normal at ' θ ' is
$y - a(\sin\theta - \theta\cos\theta) = -\cot\theta(x - a(\cos\theta + \theta\sin\theta))$
$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\cos\theta\sin\theta$
$= -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$
$\Rightarrow x\cos\theta + y\sin\theta = a$
Clearly this is an equation of straight line which is at a constant distance ' a ' from origin.

Question221

A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is
[2004]

Options:

A. $\left(\frac{9}{8}, \frac{9}{2}\right)$

B. (2,-4)

C. $\left(\frac{-9}{8}, \frac{9}{2}\right)$

D. (2,4)

Answer: A

Solution:

Solution:
Given $y^2 = 18x \Rightarrow 2y\frac{dy}{dx} = 18 \Rightarrow \frac{dy}{dx} = \frac{9}{y}$
ATQ $\frac{dy}{dt} = \frac{2dx}{dt} \Rightarrow \frac{dy}{dx} = 2$
$\Rightarrow \frac{9}{y} = 2 \Rightarrow y = \frac{9}{2}$
Putting in $y^2 = 18x \Rightarrow x = \frac{9}{8}$
$\therefore$ Required point is $\left(\frac{9}{8}, \frac{9}{2}\right)$

Question222

The normal to the curve $x = a(1 + \cos\theta)$, $y = a\sin\theta$ at ' θ ' always passes through the fixed point
[2004]

Options:

A. (a, a)

B. (0, a)

C. (0,0)

D. (a, 0)

Answer: D

Solution:

Solution:

$\Rightarrow \frac{dx}{d\theta} = -a\sin\theta$ and $y = a\sin\theta$
$\Rightarrow \frac{dy}{d\theta} = a\cos\theta$
$\therefore \frac{dy}{dx} = -\cot\theta$
∴ The slope of the normal at $\theta = \tan\theta$
∴ The equation of the normal at θ is
$y - a\sin\theta = \tan\theta(x - a - a\cos\theta)$
$\Rightarrow y\cos\theta - a\sin\theta\cos\theta = x\sin\theta - a\sin\theta - a\sin\theta\cos\theta$
$\Rightarrow x\sin\theta - y\cos\theta = a\sin\theta$
$\Rightarrow y = (x - a)\tan\theta$
which always passes through (a, 0)

Question223

A function $y = f(x)$ has a second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point (2,1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is
[2004]

Options:

A. $(x + 1)^2$

B. $(x - 1)^3$

C. $(x + 1)^3$

D. $(x - 1)^2$

Answer: B

Solution:

Solution:
$f''(x) = 6(x - 1)$. Inegrating, we get
$f'(x) = 3x^2 - 6x + c$
Slope at (2, 1) $= f'(2) = c = 3$
[∵ slope of tangent at (2,1) is 3]
$\therefore f'(x) = 3x^2 - 6x + 3 = 3(x - 1)^2$
Inegrating again, we get $f(x) = (x - 1)^3 + D$
The curve passes through (2,1)
$\Rightarrow 1 = (2 - 1)^3 + D \Rightarrow D = 0$
$\therefore f(x) = (x - 1)^3$

Question224

The real number x when added to its inverse gives the minimum value of the sum at x equal to
[2003]

Options:

A. –2

C. 1

D. –1

Answer: C

Solution:

Solution:

ATQ ,$y = x + \frac{1}{x} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2}$

For maxima. or minima.

$1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$

$\frac{d^2y}{dx^2} = \frac{2}{x^3} \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=1} = 2 > 0$

$\therefore y$ is minimum at $x = 1$

Question225

If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals
[2003]

Options:

A. $\frac{1}{2}$

B. 3

C. 1

D. 2

Answer: D

Solution:

$f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$
$f'(x) = 6x^2 - 18ax + 12a^2$
For maxima or minima.
$6x^2 - 18ax + 12a^2 = 0 \Rightarrow x^2 - 3ax + 2a^2 = 0$
$\Rightarrow x = a$ or $x = 2a$
$f''(x) = 12x - 18a$
$f''(a) = -6a < 0 \therefore f(x)$ is max. at $x = a$
$f''(2a) = 6a > 0$
$\therefore f(x)$ is min. at $x = 2a$
$\therefore p = a$ and $q = 2a$
AT Q, $p^2 = q$
$\therefore a^2 = 2a \Rightarrow a = 2$ or $a = 0$
but $a > 0$, therefore, $a = 2$.

Question226

The maximum distance from origin of a point on the curve $x = a\sin t - b\sin\left(\frac{at}{b}\right)$, $y = a\cos t - b\cos\left(\frac{at}{b}\right)$, botha, b > 0 is [2002]

Options:

A. $a - b$

B. $a + b$

C. $\sqrt{a^2 + b^2}$

D. $\sqrt{a^2 - b^2}$

Answer: B

Solution:

We know that distance of origin from
$(x, y) = \sqrt{x^2 + y^2}$

$$= \sqrt{a^2 + b^2 - 2ab\cos\left(t - \frac{at}{b}\right)}$$

$\leq \sqrt{a^2 + b^2 + 2ab}$

$\left[\cos\left(t - \frac{at}{b}\right)\}_{min} = -1\right] = a + b$

∴ Maximum distance from origin $= a + b$
